

CREATING A FLEXIBLE AND RESPONSIVE LEARNING ENVIRONMENT FOR GENERAL MATHEMATICS STUDENTS¹

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This report describes one high school teacher's attempt to change her mathematics teaching in ways that are consistent with the National Council of Teachers of Mathematics *Standards* documents (NCTM, 1989, 1991). One of the coauthors, Sandra Bethell, is the teacher; the other, Steven Kirsner, conducted regular observations in her general mathematics class, as well as student and teacher interviews. We argue that it is difficult but possible to create a classroom environment that does produce meaningful, though unpredictable, mathematical learning. We also argue that the type of teaching needed to create this environment is more likely to occur when it is accompanied by more teacher collaboration and support than was available in this case.

The Need for Change

There is an emerging consensus that mathematics teaching and learning in American schools are in need of substantial improvement. Virtually every recent national and international test reveals that a large percentage of American students are unable to perform moderately complex tasks or employ higher order thinking skills to solve mathematical problems. Students can successfully perform routine, computational skills, but are consistently unable to solve problems requiring applications of these skills (Dossey, Mullis, Lindquist, & Chambers, 1988; McKnight et al., 1987).

Many researchers attribute this situation to the traditional, didactic teaching methods and computation-dominated curricula that govern most mathematics classrooms. The teaching of computational skills comprises approximately 75 percent of the mathematics curriculum (Porter, Floden, Freeman, & Schmidt, 1988). Mechanical procedures and symbol manipulation typify mathematical learning, even in "good" classrooms (Schoenfeld, 1988). These are classrooms where students embark on "wild-goose chases" in search of right answers, with little regard for how meaningful these answers might be (Erlwanger, 1975). For these students, mathematics learning means memorizing rule-bound algorithmic skills and procedures that are devoid of context or meaning. This kind of mathematics learning reflects the trivialized, superficial learning that prevails in American classrooms more generally, regardless of subject matter (McNeil, 1986; Sedlak, Wheeler, Pullin, & Cusick, 1986).

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Researchers who document trivialized learning in mathematics classrooms describe a teaching pattern that is firmly entrenched by the upper elementary grades (Mathematical Sciences Education Board, 1990; Romberg & Carpenter, 1986; Schoenfeld, in press; Schram, Wilcox, Lanier, & Lappan, 1988; Stodolosky, 1988). Teachers typically begin classes by reviewing the preceding day's assignment. They then introduce the present day's lesson and explain a few sample problems. (The demonstration of these examples often constitutes the entire lesson). During the remainder of the class, students work individually on assigned problems very similar to the examples the teacher performed during the lesson. Those students who attend to the assignment may receive help from the teacher if they request it. Only after students have spent time practicing a computational skill are they typically exposed to application "problems"—usually word problems. This pattern is repeated virtually daily (Romberg, 1992).

This state of teaching practice is consistent with a vision of mathematics learning that has guided the school mathematics curriculum since the 19th century. Thomas Romberg (1988) refers to this computation-dominated curriculum as "shopkeeper" arithmetic. Twentieth-century reformers have periodically recognized the outdated nature of this curriculum and have offered newer visions of school mathematics, visions that are more consistent with the modern, technological era. These newer visions of mathematics teaching and learning are inconsistent with traditional practice. However, the traditional practice and shopkeeper curriculum that served the older vision have proved extremely persistent. Newer visions of mathematics teaching and learning in this century have been repeatedly vitiated by the staying power of traditional practice and shopkeeper arithmetic.

The NCTM *Standards* Vision of Mathematics Education

Contemporary efforts to reform mathematics teaching and learning also begin with the premise that shopkeeper arithmetic and traditional practice should be replaced by a new vision of mathematics teaching, learning, and curriculum. This modern vision is often cast in such terms as teaching for understanding, conceptual understanding, problem solving, higher order thinking skills, or quantitative and spatial reasoning. There is a high degree of consensus among scholars, researchers, and curriculum leaders in the mathematics education community that the most coherent and comprehensive new vision of school mathematics is contained in the National Council of Teachers of Mathematics *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) and portrayed more vividly (in vignette form) in the companion document, *Professional Standards for Teaching Mathematics* (NCTM, 1991). Both *Standards* documents have been endorsed by the U.S. Department of Education (Kirsner, 1990), the National Education Goals Panel,

and the National Council on Testing and Standards, as well as by many professional education organizations.

The most striking, and compelling, aspect of the *Standards* documents (NCTM, 1989, 1991) is how their vision of mathematics teaching and learning contrasts dramatically with the prevailing tradition-oriented practice in mathematics curriculum, instruction, and assessment. The nature of mathematics teaching, mathematics learning, and mathematical knowledge described in the *Standards* depart significantly from the broadcasting method of teaching (National Research Council, 1989), trivialized and surrogate learning (Sedlak et al., 1986), and knowledge as facts (Cohen, 1988) that characterize traditional practice. A major premise of the *Standards* is that students are active learners. Young children enter school with a natural curiosity about and intuitive knowledge of numbers and mathematics. They do not passively absorb new knowledge but actively build on their intuitive, informal knowledge as they construct new meanings and understandings (Baroody, 1987; Ginsburg, 1977; Resnick, 1987). Consequently, the *Standards* portray classrooms as places where students are regularly involved in actually solving interesting mathematical problems, both as individual and group endeavors.

The *Standards* call for students to "examine," "transform," "apply," "prove," "communicate," and "solve" mathematical problems and concepts. For this to occur, teachers must assume the more difficult but more rewarding role of managing classrooms where students are actively engaged in such activities as making presentations, conducting experiments, working in groups, and participating in discussions, as well as working individually (Kirsner, 1990).

This report documents one teacher's attempt to create in her 10th-grade classroom such an environment. We will pay particular attention to student learning, as well as to the conditions that promote or limit meaningful mathematical learning among students. Data collection included classroom observations, observations of formal and informal professional meetings in which the teacher participated, interviews of the teacher, and interviews with four students over a four-month period. The high school in which the data were collected was in the second semester of its second year of being a "professional development school."

Professional Development Schools

Michigan State University has helped establish Professional Development Schools where MSU teacher educators and researchers work with the PDS staff in order to begin to create a climate that supports teaching for understanding. For example, PDS faculties attempt to replace a working environment traditionally characterized by isolation with one of collaboration. PDS faculty members are encouraged to teach with and observe colleagues, work with prospective teachers, and participate with university staff in research projects. To help establish norms of intellectual inquiry,

PDS teachers have opportunities to examine and discuss research reports and reform proposals, to reflect on their practice, and to participate in research activities. These activities contrast with traditional school norms (Lortie, 1975).

In short, PDS sites attempt to reduce or eliminate those contextual features of schooling that have been identified in the teacher education and school organization literature as barriers to the successful implementation of teaching for understanding. They attempt to alter fundamentally the "older and much more traditional mathematical and pedagogical structure" (Cohen & Ball, 1990b, p. 352) that is proving to be such a serious impediment to the successful implementation of the California mathematics education reforms (California State Department of Education, 1985). They represent a serious attempt to realize the *Standards* (NCTM, 1989, 1991) vision of mathematics education through an ambitious, systemic school restructuring implementation strategy.

Obstacles to Reform

While it is reasonable to expect that the PDS will make some progress toward successful implementation of the *Standards* (NCTM, 1989, 1991), it is certain that there will be implementation difficulties. As David Cohen (1988) has noted, many educators fail to appreciate how difficult a task it is to teach for understanding on a regular basis. Although the *Standards* documents present a vision of mathematics teaching and learning, there is no clear road map (nor can there be) of how individual teachers might achieve that vision. A dearth of compatible curriculum materials, a rigid school structure, and students who have internalized and come to expect traditional teaching and learning are some of the obstacles likely to confront teachers who actively attempt to teach according to the *Standards*.

Moreover, Linda Darling-Hammond (1990) refers to "policy collisions"—that is, previous or extant policy conditions or initiatives that directly contravene, or at least seriously inhibit, the successful implementation of new policy reforms such as the *Standards* vision of mathematics education. For example, Cohen & Ball (1990a) note that today's policies, which intend to promote teaching for understanding, come on the heels of recent policies that encouraged traditional, didactic teaching. For the teachers they studied, "the result [of this policy collision] was a curious blend of direct instruction and teaching for understanding" (p. 252). Also, the content and administration of mandated, standardized mathematics tests present potential policy collisions. The National Center for Research in Mathematical Sciences Education has documented the deleterious effects of these tests on teaching for understanding (Romberg, Zarinnia, & Williams, 1989). The classroom environment we investigated was not immune from similar constraints.

Students' predispositions toward school and learning presented a particularly difficult obstacle for Sandy to overcome. One of Sandy's objectives was to encourage and empower students

to construct and communicate their own mathematical knowledge. However, there were significant practical difficulties involved in orchestrating classroom activities and by discussions so as to facilitate this kind of mathematical learning. By not using a textbook; by relying on discussions, group work, and journal writing; and by deemphasizing computation; Sandy was in effect "changing the rules" of learning and teaching that her students had internalized for years. The students' resistance to these new rules was, albeit understandable, a persistent constraint to the kind of learning Sandy tried to foster.

Ironically, whereas researchers (e.g., Lanier & Sedlak, 1989; Schoenfeld, 1988; Sedlak et al., 1986) characterize traditional school learning as a form of gamesmanship, it was Sandy's nontraditional teaching that her students perceived as a game, and a demeaning one at that, especially since they were unfamiliar with the new rules. In spite of this resistance, which seemed to diminish over time, students at times were able to demonstrate important aspects of meaningful mathematical learning. Not only had these students internalized a kind of school and mathematics learning antithetical to that espoused by Sandy and the *Standards* (NCTM, 1989, 1991), but they had learned that they were not supposed to succeed in mathematics classes. If they were expected to be successful, they certainly would not have been assigned to general mathematics, where they typically would be expected to rehearse the same low-level computations for the third, fourth, or fifth consecutive year. Sandy also confronted difficulties as she tried to implement the *Standards* because of (a) her self-perceived isolation from other mathematics teachers, which Sandy attributed to their perceptions that she held radical views on teaching; (b) a lack of curriculum materials that would support the kind of practice she engaged in; and (c) her own subject matter limitations.

The Site: Park High School³

Park High School is located in a predominantly white working class and middle class community. The school serves nearly 1,000 students in grades 10-12. During the data collection period, it was in its second full year of PDS activities, with 25 of its 55 faculty members and two of three administrators (principal and one assistant principal) participating in PDS activities. Sixteen MSU professionals—8 faculty and 8 advanced graduate students—had a 25-50 percent assignment in the PDS. During its first year as a PDS, participants worked on exploring current practices and designing interventions aimed at enhancing both high school and teacher education students' learning and understandings. Park PDS participants report that they are

³The name of the school and names of students are pseudonyms. The description of the site is taken from the *Educational Extension Service Third-Year Plan*, Michigan State University, College of Education, 1990, pp. 49-60.

- Designing and implementing curriculum units around a conceptual model of teaching and learning
- Experimenting with alternate ways of grouping students in order to meet students' diverse needs, abilities, and future aims
- Conducting inquiries about teaching practice and ways to help students understand subject matter at a conceptual level
- Conducting interdisciplinary research and/or development

The subject-specific project that is of major concern to our report is the Mathematics Project. It is "aimed at helping students develop conceptual understandings of mathematics, including practical math, statistics and probability, algebra, and enhance self-efficacy and appreciation of mathematics" (Michigan State University, College of Education, 1990, p. 55). Four questions guide the Project: (a) What are the academic, social, and personal constraints that confront the teacher who chooses to work for conceptual change with practical mathematics students? (b) How can a course on statistics and probability be made meaningful for *all* high school students? (c) What and how can strategies be implemented that enhance self-efficacy and keep mathematical studies as a future option for students? (d) How does the revision of curriculum and use of technology affect students' knowledge and appreciation of mathematics? Project participants met regularly on Wednesday mornings to address these questions.⁴

The Teacher—Sandra Bethell

Sandy is a high school mathematics and foreign language teacher who has taught for four years and studied education policy and mathematics education for three years at the graduate level at Michigan State University. She is dedicated to teaching mathematics for understanding, believing that mathematical knowledge can be meaningful and empowering for high school students. Sandy believes strongly that tracking has denied many students access to high-status knowledge. This includes those students who have been placed in the special education, general, or vocational tracks. Believing that all students are capable and deserve to learn powerful mathematics, Sandy volunteered to teach a general mathematics class of 16 students, 10 of whom were labeled "special education" at the time they were in class or before. She attempted to create a learning community in which students were encouraged to engage actively in mathematical problem solving and to reason mathematically, orally and by writing, within that community. One of her objectives was

⁴The Mathematics Study Group met weekly for staff development activities at which time Park High School students did not attend classes until 11:30 a.m.

to encourage and empower students to construct and communicate their own mathematical knowledge.

Before her graduate studies in teacher education and mathematics education, Sandy taught fairly traditionally. Her fundamental beliefs about teaching, learning, curriculum, and equity changed significantly during her graduate study. This partially explains the isolation she felt upon returning to teach at Park High during the second year of PDS activities. Most of her colleagues were just being exposed to these different ideas about teaching and learning. Changing one's fundamental beliefs takes time; changing one's practice to reflect the changed beliefs takes more time. Although many teachers' beliefs were being challenged, and some were changing, few, if any, teachers seemed to be changing their practice significantly. Sandy felt that her attempts to alter her teaching practice radically set her apart from her colleagues. This accounts for some of the isolation and lack of support that Sandy perceived.

Sandy's Math Class

All of the students in Sandy's class, as is the case for most senior high school students assigned to general math, had encountered various degrees of failure and frustration during their previous school mathematics experiences. Sandy was especially intent on instilling in her students a sense of efficacy in their abilities to do mathematics. Her conception of doing mathematics corresponds to the first four standards of the *Curriculum and Evaluation Standards*—mathematics as problem solving, as communication, as reasoning, and mathematical connections (NCTM, 1989). Above all, she wanted all of her students to "make sense of mathematical ideas."

Sandy believes that all students are capable of learning meaningful mathematics if they are given the opportunity. She believes that her general mathematics students, for one reason or another have never been presented with such an opportunity. By having been assigned to low mathematics tracks, these students have been denied access to high-status knowledge as they received the message that they are not very intelligent. As they progressed through their schooling, the achievement gap between them and their peers in higher tracks most likely widened, further reinforcing their low self-concepts⁵ (Oakes, 1985). Senior high school students taking general mathematics are often presented a curriculum of the same low-level computational skills that they have had literally year after year since fourth or fifth grade. One of Sandy's most important goals was, and continues to be, to give these students opportunities to learn important mathematics.

In order to create an environment where these students could be successful, Sandy employed a rather eclectic blend of teaching activities. Within the context of the *Curriculum and Evaluation*

⁵This is exacerbated for special education students who have been labeled and removed from regular education classrooms, as we will see when we examine one student's interview data.

Standards (NCTM, 1989) and the *Professional Teaching Standards* (NCTM, 1991) are a wide variety of teaching activities. Among mathematics educators, there are differences, largely in degrees of emphasis, among different teaching strategies. Whereas some might advocate a heavy reliance on manipulatives, others give greater emphasis to student discussion and communication about mathematical ideas. Yet others emphasize group work and cooperative learning. Traditional teaching, of course, relies on teacher demonstrations and individual student seatwork. Our field notes reveal a broad approach that emphasized each of these types of teaching activities roughly equally. Sandy did not use a textbook in her class. She relied on materials she had collected from course work and workshops; wrote her own activities, worksheets, and tests; and occasionally consulted with other math educators.

During much of the data collection period, students were engaged in a unit on probability. Although probability and statistics are stressed throughout all grade levels by the *Standards* documents, hardly any K-12 students are exposed to this important branch of mathematics (Shaughnessy, 1992). According to Shaughnessy, "the impediments to effective teaching of probability are the same ones that hinder effective implementation of problem solving in our schools . . . in fact, *teaching stochastics* [i.e., probability and statistics] *is teaching problem solving*" (pp. 466-67).

The entire unit lasted approximately 10 weeks. Since the prescribed district curriculum for general mathematics did not include probability, Sandy justified this unit on the grounds that it helped students learn to compute with fractions, which did occupy a central place in the prescribed curriculum. This is one way that Sandy circumvented an important policy collision—"I just framed the problem-solving curriculum that I wanted to teach in the context of helping students learn the computational skills that the prescribed curriculum dictated." Students explored the meaning of probability, the concepts of experimental and theoretical probability, the meaning and usefulness of simulating experiments, and the concepts of independent and dependent probability. Sandy relied on her multifaceted repertoire of teaching strategies throughout the unit. Two problems in particular are illustrative.

We refer to each of the classes we describe as "problems." This reflects an important aspect of the *Standards*—that problem solving should be an essential component of learning and instruction. In each class, Sandy presents a problem and gives students opportunities to work together on them. At times, students work as Sandy answers questions from individuals or small groups. At other times, she guides a whole-class discussion. Often students go to the board or overhead to explain a point or ask a question.

Although we believe that each class portrays an environment that supports student learning, in neither case do all students demonstrate such learning. However, some of the important

mathematical learning that did occur was not immediately evident. It did not surface until we interviewed students weeks after each problem was explored. We will also discuss Sandy's perception that more frequent opportunities to discuss with colleagues her teaching and her students' learning would have helped her considerably.

Problem 1: Driving While Sober or Intoxicated

The first problem had to do with the probability of having an automobile accident, given that the driver is either sober or intoxicated. (As is the case in many American high schools, drinking is a significant problem at Park High School; in fact, Sandy knew that many of her general mathematics students, most of whom drove cars, drank alcohol regularly and often excessively.) Sandy presented these statistics:

- At any one time, 2% of the driving age population are intoxicated
- At any one time, 98% of the driving age population are sober
- The probability of having an automobile accident given one is intoxicated is .00045
- The probability of having an automobile accident given one is sober is .00016
- There are 180,000,000 Americans with drivers' licenses (this is only an estimate)

Sandy then gave her students opportunities to work in pairs and in groups over two class periods to make sense of the data and draw some conclusions. Our classroom observation occurred on the second day of this activity. Halfway through this class, students were asked to discuss their work and their thinking. A student explained that he and his partner had multiplied $.02 \times .00045$ and $.98 \times .00016$. The student went on to present the products, .000009 for intoxicated drivers who have accidents and .0001568 for sober drivers who have accidents. He explained that .000009 is approximately equivalent to $0/1,000$, $0/10,000$, $1/100,000$, $10/1,000,000$, $100/10,000,000$. Similarly, .0001568 is equivalent to $1,568/10,000,000$.

He then multiplied these products by the number of people of driving age, estimated at 180,000,000. He portrayed these products as the number of sober people in the driving population who have accidents ($.0001568 \times 180,000,000$) and the number of intoxicated drivers who have accidents ($.000009 \times 180,000,000$). These products yield 1,620 out of 180,000,000 drivers who are intoxicated *and* have accidents and 28,224 out of 180,000,000 who are sober *and* have accidents.

After the student explained this comparison in more detail in response to another student's having expressed confusion, a third student asked a question: "Does that mean that we are better off drinking and driving since many more people are sober and have accidents?" A discussion ensued, but the end of the class was imminent.

Steve, who was observing the class, and who had become a part-time participant in the class, volunteered that the students might want to think about the ratio of sober to intoxicated drivers independently of how many out of each group have accidents. They might then make some conclusions about the actual number of sober and intoxicated people who have accidents. Although there are 49 times as many people who are sober than intoxicated at a given time (i.e., $.98/.02$), there are only about 17 times as many sober people than intoxicated people who have accidents (i.e., $28,224/1,620$). (The $.98/.02$ ratio divided by the ratio $.00045/.00016$ equals the ratio $28,224/1620$, or $17.4222 \dots$)

Immediately after the class, Steve interviewed Sandy. Her immediate reaction to the class was that there was a great deal of confusion among the students. She realized that the nature of the problem was troublesome.

I'm pretty sure most of them thought \dots your chances are less in getting in an accident [if you are intoxicated] \dots except for about three. \dots I think that about three of them got [it that the intoxicated ones would] get in an accident. And I bet, even the rest of them, even now, if they had it again they would guess that it would be the drunk [who had the lower probability of having an accident].

She went on to say that, "We'll just have to revisit those later. I want to try it with different numbers \dots simpler numbers that are more reasonable to them."

This conversation indicates that Sandy analyzes student learning, the mathematical tasks, and the environment in order to make ongoing instructional decisions. Unfortunately, Sandy did not feel that she had regular opportunities to discuss her analysis with her colleagues, as she did here in the context of a formal interview. (Even in the absence of an interview, she did have opportunities to discuss such issues with Steve when he was present for an observation since her planning period followed the general mathematics class—it would have been ideal if such opportunities were available more regularly.)

In retrospect, when we look at the interview transcripts and reflect on the class and the interview, we realize that Steve, who tried to clarify some student confusion before class ended, was fearful that students would leave class confused. Sandy was not as concerned about this. She thinks that part of teaching and learning for understanding necessarily implies that student misconceptions might linger. She was more concerned with the fact that the cumbersome nature of the decimal

numbers may have interfered with students' trying to make sense of the problems. Therefore, she immediately thought of using simpler numbers when revisiting this topic in order to help students correct their own misconceptions about whether to drive when intoxicated. (At no time did Sandy fear that the students' misunderstanding meant that they would increase their frequency of driving when they were drinking; nor was this the basis for Steve's not wanting them to leave class with misconceptions.)

We also realize that, even though many students did not seem to be reasoning about the problem as ably as the student who explained how he made sense of it, most students were able to handle computations with fractions, even though the unfamiliarity with such small fractions may have detracted from some students' overall understanding of the probabilistic nature of the problem. They were indeed learning computational and reasoning skills related to fractions in the context of a problem solving situation. Earlier we noted that Sandy was able to circumvent a policy collision by justifying her problem-oriented probability unit on the grounds that it would improve computational skills as required by the district curriculum. Ironically, we doubt that the facility with fractions that some students demonstrated in the context of solving a meaningful probability problem would have been evident had Sandy confined herself to teaching fractions traditionally.

This assertion is supported by Romberg (1992): "Thus, present strategies for teaching mathematics by first teaching skills and then exposing students to stylized application problems need to be reversed; concepts and skills should emerge from experience with problematic situations" (p. 37). We also understood, and continue to understand as we write about the problem, that this is complex subject matter. We talk to each other about the mathematical meaning of the problem, about what we hope students might learn from it, and how might we better represent this or a similar problem.⁶

As we look at field notes from this and other classes, we are confident that Sandy attends to the *Professional Teaching Standards* (NCTM, 1991) framework of four dimensions that emphasize the important decisions that teachers need to make as they teach for understanding:

- Setting goals and selecting or creating mathematical *tasks* to help students achieve these goals
- Stimulating and managing classroom *discourse* so that both the students and the teacher are clearer about what is being learned
- Creating a classroom *environment* to support teaching and learning mathematics

⁶These are the kinds of discussions that Sandy would like to have had with her mathematics colleagues on a regular basis.

- *Analyzing* student learning, the mathematical tasks, and the environment in order to make ongoing instructional decisions

Unfortunately, Sandy does not feel that she has an abundance of resources at her disposal as she attends to these dimensions. Although Park High School teachers regularly talk about changing practice and teaching for understanding, Sandy did not feel that there were many of her colleagues who regularly thought about how to represent problem situations. Every other mathematics teacher relied on a textbook. She could not imagine one of them who spent an entire class period exploring one or two problems. Even in the regular mathematics study group meetings, teachers rarely discussed or solicited advice about their own teaching. (This did change during the last month or two of the school year, largely due to Steve's intervention, but this occurred after the probability unit.) Given Sandy's nontraditional views of teaching mathematics, she perceived that other teachers saw her as an outsider, which made her more reluctant to raise issues related to her own teaching.

Moreover, given the time constraints of a regular teaching load, Sandy did not have much of an opportunity to consult mathematics educators from the university. She considered this to be an important source of meaningful, challenging problems. She felt strongly that her own college mathematics courses had not prepared her to think about mathematics in interesting ways. She was able to consult with Steve on occasion and with a special education teacher who sometimes visited her general math class and who was also a doctoral student in teacher education, although not in mathematics education. Sandy was frustrated by what she perceived as a lack of support for her own learning about teaching mathematics for understanding. Although she is confident that she attends carefully to the four dimensions of the *Professional Teaching Standards* (NCTM, 1991), she is equally confident that increased collaboration and support would greatly improve her and her students' learning.

Problem 2: Monty's Problem⁷

Steve introduced this problem to Sandy because it had recently been presented to him by a university colleague as an interesting problem with a counterintuitive solution. Although Steve was not suggesting it for use in her class, Sandy thought it was so interesting that she immediately thought to pose it to her students, who were in the midst of their probability unit. Shaughnessy and Dick (1991) frame the problem this way:

⁷The name of this problem comes from Shaughnessy and Dick (1991), although we were not aware of their article until after the problem was used in class.

During a certain game show, contestants are shown three doors. One of the doors has a big prize behind it, and the other two have junk behind them. The contestants are asked to pick a door, which remains closed to them. Then the game show host, Monty, opens one of the other two doors and reveals the contents to the contestant. Monty always chooses a door with a gag gift behind it. The contestants are then given the option to stick with their original choice or switch to the unopened door. (p. 252)

The problem is: If you were a contestant, would you stick with your original door, switch to the other door, or does it even matter what you decide?

When Steve and Sandy first discussed this problem, Sandy refused to accept the conclusion that the contestant can double his/her chance by switching to the other door. She suggested that it doesn't matter whether one stays or switches—after learning that one door has been eliminated as a winning one, there remain two equally likely chances, the original choice or the remaining door. Her response was typical of "people across the whole range of mathematical expertise, from novices to teachers to research mathematicians" (Shaughnessy & Dick, 1991, p. 252). She immediately decided to "see what my students think."

When Sandy posed the problem to her students, she told them that she and Steve had different opinions about the most advantageous strategy for the contestants. The students predictably agreed with Sandy, who asked them how they could prove to Steve that he was wrong when he made his next visit. A number of students responded that they could simulate the game. They proceeded to simulate the problem in pairs. It was not long before students observed that contestants were winning more often when they switched than when they stayed with their original selection. Not totally convinced, the class decided to alter the manner in which the simulations were conducted.

They decided to conduct a larger number of simulations and to do them more systematically. Sandy kept track of the simulation responses that were generated in a whole-class format. Each pair would give 20 of the same responses (i.e., either stay or switch) while others in the class recorded the results. By the end of the class, students were convinced that it is advantageous to switch doors because this strategy wins the prize twice as often as the other strategy. They were also perplexed, and expressed curiosity about why this was the case.

The next time Steve came to observe class, the students were eager to present their findings. They also wanted an explanation for their counterintuitive findings. Steve offered different explanations for the clear advantages of the switch strategy. He left class not quite sure whether students understood and/or accepted his explanations. However, it did seem clear that students' simulations were convincing evidence for them that the switch strategy was the preferred one. Over the next few days, other adults (e.g., a special education teacher and a student teacher) came to

discuss the problem with the students. In each case, the students explained that the switch strategy was the preferred one, and a discussion ensued about why this was the case.

Although most students had trouble making sense of *why* the switch strategy was advantageous, they clearly accepted that it was. They demonstrated an understanding of the concept, strategy, and usefulness of simulating experiments. The understanding was such that they accepted the evidence yielded by the simulations and rejected their original, strongly held guess about a desired strategy. Moreover, they conducted the simulations and discussed the evidence in an environment that closely approximates the vision of the *Professional Teaching Standards*:

The teacher of mathematics should create a learning environment that fosters the development of each student's mathematical power by—

- providing and structuring the time necessary to explore sound mathematics and grapple with significant ideas and problems; . . .
- providing a context that encourages the development of mathematical skill and proficiency;
- respecting and valuing students' ideas, ways of thinking, and mathematical dispositions;

and by consistently expecting and encouraging students to—

- work independently or collaboratively to make sense of mathematics;
- take intellectual risks by raising questions and formulating conjectures;
- display a sense of mathematical competence by validating and supporting ideas with mathematical argument. (NCTM, 1991, p. 57)

Evidence from Student Interviews

Although we have portrayed Sandy's classroom as an environment that encouraged and promoted meaningful mathematical learning, our evidence of student learning from interviews at the end of the year is mixed with respect to what students learned. We present interview data from two students as illustrative of the complex and unpredictable learning that occurred. What we learned further reinforced Sandy's use of student interviews as assessment tools—assessment of her instruction as well as of student learning.

Penelope

Penelope is a sophomore and a special education student. Her interview data reveal a level of mathematical learning surprising to us. Although she seemed disengaged from classroom activities, and often expressed verbally that she was not smart enough to understand the subject matter, Penelope's interview responses demonstrate a relatively sophisticated understanding of probability. Her case illustrates how unpredictable, and difficult to discern, student learning often is. Excerpts of the interview follow with Steve (SK). Penelope (P) responds by talking about how she had never before been given an opportunity to study important mathematics and about how different and positive this math class has been for her:

SK: OK. Can you tell me a little bit about this year's math class. Is it a lot different from or different at all from previous math classes and if so how? What's it been like?

P: Yeah. It's different. I like this math class. And I like Ms. Bethell. She's a good teacher. It's really good for my other math class because the school I used to go to, when they had special ed. they always pulled us out of the class. And it made us feel a little bit, kind of, not so particular. We don't feel self-confidence in ourself. We feel a little bit [more] stupid than other people. They just pull us out. And here they don't have to pull us out. It's not like we're in a special class . . . and plus, I get a better grade in this math class than I do last year.

Even though it's basically the same, it's just that I learn more here than I did over there because over there all I learned is all that conditions. Here I learned geometry, or how to do fractions, which I had no idea how to do and I didn't know how to do geometry. And I learned probability and all this stuff that I never learned when I was in special ed. So all the stuff I learned was basic over there in special ed. in that other school. And they didn't show me probability, geometry, or fractions, or how to use fractions or simplify them or cross-divide. They didn't show me all that. So therefore, I was behind.

But when I came here, they put me up where I was. . . . I always thought I was stupid in mathematics. And Ms. Bethell thinks I'm very smart in it. I always felt bad because I didn't know all the stuff that everybody else knew, and now I know that stuff that everybody else knows now. And I'm really enjoying being in this class. And it's just did me good. And now I don't feel stupid. I feel kind of smart. I mean because I know stuff that my aunt don't know and she wants me to come over and teach her how to do this stuff. And when I take pre-algebra, I'm going to take Algebra 1, I mean, next year with Ms. Bethell, because we heard she might be teaching Algebra 1 again, and she wants me to come over and teach, help her learn stuff like that. So I kind of feel great about myself.

Although it seems that Penelope recognizes that she had long been denied access to high status knowledge in mathematics, she understandably still holds to a traditional utilitarian view of the importance of learning (even high status) mathematics:

SK: Why are you taking mathematics?

P: Because I think that I'll be using mathematics when I'm older and—it would be practically in everything I do: balancing my check, my tax, and also a secretary doing those, what my job is supposed to be like. Figuring out checks somewhere and the accounts and stuff and how much money they have in that company. Or anything, actually everything you do is tied to mathematics. Figuring out your shopping list and stuff. I think everything is mathematical.

SK: Well, that's interesting. Do you want to say anything more about that, about everything being mathematical? Because I think that's a very interesting idea.

P: Yeah. See, even when you drive your car you have numbers and gauges and stuff and you have to know what it means and stuff. And that's mathematical. Like being an astronaut, you have to know science, math, and also, in armed forces you got to know your math. A lot of math is in there. You know like anything like broken, like the fan belt or anything you have to know the sides what they're about, like how many inches, what halves and stuff. You have to know that as being a mechanic and stuff.

In contrast to the traditional teaching represented above, Penelope indicated that students learn from each other as well as from the teacher in this math class, which supports our assertion that Sandy's students did mathematics in a learning community, where learning was a shared responsibility and endeavor:

SK: What's that been like to work in groups instead of what you're used to?

P: Well, first of all, when I worked in, when I didn't work in groups, it was harder to get to know people. It was hard to do math because the people who know how to do it we could learn from each other's ideas. But we didn't do it over there. *We just worked separate and we actually didn't learn practically anything but what we learned from the teacher. Here we learn from everybody. We learn how they do it, how they understand it and we share our ideas with each other.* (italics added)

She also demonstrated a sophisticated perspective on how one learns mathematics:

- SK: Is it something that you just have to know and has to already be in your head? . . .
- P: Something you learn, too. Because you learn more. I mean even, like my teacher showed me, I don't know what it's called, but hands-on experience, you learn that too, as well as you go on. And you go more. Each year you grow. You get older and stuff.
- SK: So do you think that what you're learning in this mathematics class will allow you to keep learning mathematics and figure things out for yourself.
- P: Yep. And I won't be ashamed to ask anybody for help if I'm stuck.
- SK: Well, that's great. Is that something that you think that you acquired this year?
- P: Yeah. Because if I'm stuck and I don't know the question I admit that I don't know it. Or I . . .
- SK: And whom do you ask?
- P: I ask the teacher or when they go, "Do you understand?" out loud, you know when they ask you if you understand, I just say, "No, I don't understand it. And explain."
- SK: Is there anyone else you would ask beside the teacher?
- P: A student, my sister [who is also in the class], or anybody else who's sitting around me who can help me.

The seriousness and sophistication with which Penelope was responding was surprising to us. In class she regularly acts as if she were disengaged from virtually all academic tasks. She often expresses that she is "too stupid to learn." She spends considerable time walking around the room and often asks to put her head down because she has a headache. Yet, she made the following responses in the interview situation:

- SK: Can you tell me either what probability is about or give me an example or tell me why it might be important to study probability?
- P: Well. I have it written down what probability is about. See if I can find it.
- SK: Sure. Or if you can think about an example.
- P: Probability. Ah. Here it is. It's a chance that something will happen. It is expressed as a ratio. Numbers of sequences, numbers of total numbers, you know. Like if you flip a quarter you can, you have either a chance of getting a heads or a tails. The

probability is, is one half. Because there's two sides, one with a heads and one with a tails. So actually there's a one half. You can get one heads if you flip and one tails if you flip so there's actually one half. That's the probability of getting it. Or you can get. . . . This is where I get stuck sometimes. With that one thing that you showed us. Like that ABC thing. You know.

SK: Oh yeah. Where there are three doors: A, B, and C. And there's a prize behind one of them.

P: Yep. There could be—I don't really know quite what you said, but I remember there was a one-third chance of being answered right. And so there's a two-thirds chance if you switch.

SK: Here was the problem. The problem was ABC. And say behind one of these is a \$1,000 bill. And you pick one.

P: A

SK: And then I tell you it's not behind this one [C]. Should you stay with A or should you switch to B?

P: I'll switch because you get more of a chance of getting the answer right if you switch. Which my teacher, at first she thought that it'd be the same. You'd get a half and half chance. But it isn't because you started out with three. And then when they ask you that it wasn't this one, you could change. You have the chance of changing, or not changing. And so you have a better chance of changing than you do by staying.

SK: Can you explain why?

P: I—Oh, that's harder.

(This interview occurred approximately five weeks after Monty's problem was discussed in class.)

SK: OK. Let's see. Is there anything that—tell me what you've learned—and this is the last question. Can you talk about anything that you've learned in math class this year that you think is important?

P: I learned the probability. I thought that was important because everything you do is going to be a probability. Like how much would you spend—I mean like how much more would you save on a bus, a plane or a train or just by car. How much would you save. And the probability is like—what chance—no, oh I know. What is the safest ride? That's it. That's what I was thinking about. What is the safest

ride. The probability would be if we used that, the probabilities would be—the safest thing . . .

SK: The safest form of transportation. How about that?

P: Yeah. The safest form of transportation would be like flying, riding a train, riding the bus or car. But I think planes are more safer, even though there's a lot of cautions.

SK: And why do you think planes are the safest?

P: Because they are, because last time I ever heard, they add all the plane crashes, all the train crashes, all the car crashes, and all the bus crashes. But I think buses and cars go together because they are both on the road together, right. So I just say all the car crashes and they found out with all that, and on the probability of getting that because you can't add exactly the whole plane crashes, because there is a lot of them and a lot of people riding it, so they say the safest way, the safest way is riding a plane. Because you can't actually crash into something unless you run out of gas or something. Or . . .

SK: Why do you think that the safest is a plane? What do you think that's based on?

P: Theoretical. Theoretical because. . . . Maybe theoretical and experimental. I say both.

SK: And how would it be experimental?

P: Experimental by figuring out how many plane crashes go and theoretical is like they did it before they found out.

SK: So based on how many actual plane crashes, you're saying that there aren't as many plane crashes, not as many people died and that's why it's safer.

P: Yeah. That's what I heard. But since then there's been a lot of plane crashes. It might be different now. Because it was a long time ago. They haven't done updates on this stuff anymore.

Shaughnessy (1992) uses four categories to characterize people's stochastic conceptions: Non-statistical, naive-statistical, emergent-statistical, and pragmatic-statistical. Some of each category's indicators include the following:

1. Non-statistical: responses based on beliefs, . . . no attention to or awareness of chance or random events.

2. Naive-statistical: mostly experientially based . . . responses; some understanding of chance or random events.
3. Emergent-statistical: ability to apply normative models to simple problems; recognition that there is a difference between intuitive beliefs and a mathematized model.
4. Pragmatic-statistical: an in-depth understanding of mathematical models of chance. (p. 435)

We believe that Penelope, who had never before studied probability, and who refers to herself as "stupid," is a student in the emergent-statistical stage. This is how Shaughnessy (1992) describes such students:

Students in the emergent-statistical stage are under the influence of didactical interventions, and their conceptions are changing. They are beginning to see the difference between degree of belief, and a mathematical model of a sample space. They are becoming able to apply normative models to an ever widening range of stochastic settings, but initially they may still be subject to falling back upon the familiar, causal or heuristic explanations when confronted with an unfamiliar type of task. Their normative conceptions are developing but still unstable. (p. 486)

We believe that Penelope's interview demonstrates that she falls into this category. At the very least, she has clearly moved out of the nonstatistical category. This is not an insignificant effect for a 10-week intervention, and it is unlikely to have occurred in the context of traditional instruction in probability (Shaughnessy, 1992).

Gene

Gene is a very personable sophomore who also has been labeled a special education student. Throughout the year he has been noticeably disengaged from academic learning. He jokes a lot in class and rarely asks questions or participates in class discussions. At least, this was the case through the middle of May.

On May 14, we had our first interview. Gene's responses seemed consistent with his behaviors in class. However, shortly after this interview, Gene astounded us by becoming enthusiastically engaged about the mathematical content of a geometry unit. He observed a pattern about polygons, formulated a hypothesis, and tried assiduously to make sense of his observations during the last three weeks of class. What came to be discussed as "Gene's theorem" became the subject matter of at least four class discussions and demonstrations.

Although we cannot explain this change in Gene's attitude and behavior with any degree of certainty, we do assert that the classroom environment was (a) supportive of this type of mathematical inquiry and reasoning and (b) flexible enough to give Gene multiple opportunities to pursue his theorem and refine his thinking with the entire class. We feel that this is consistent with many of the content and pedagogical goals of the *Standards* (NCTM, 1989, 1991) documents. The classroom environment reflected a learning community who were trying to make sense of the hypothesis of one of its members.

We begin with excerpts from the May 14 interview, during which Gene expresses a preference for traditional mathematics instruction, disdains group work, yet expresses the view that the mathematics learning that Sandy is trying to promote is important. We then move to an interview that occurred after Gene invented his theorem. From the first interview:

SK: OK. I'm going to start by asking you just to tell me about, a little about this year's mathematics class. Specifically, does it differ from other math classes you have and if so, how? If not, why is it the same?

G: It's different. This is the first year we haven't had a math book. Like recent years, you know, they just give you a math book, and they'll tell you go home and do this page and that page. This whole year we haven't had a book. We just do, like we had that packet that took us a while, that probability packet, and then there's others. We did packets and mostly in-class work to see how we were doing. She was trying to study this year. So we didn't use a book.

SK: And what were your—did you have any reactions to it?

G: I would rather have a book.

SK: Really?

G: Yeah. Because then if you were, you know, you didn't understand something at home you had the book. You could just go look at the example and go through previous work that you're trying to understand something. You can't do that.

When I asked him about working in groups, he was less than enthusiastic:

SK: What about—I've noticed that a lot of times in class you work in groups. What's that been like? Has it helped you learn mathematics? Or what are your thoughts on that?

G: I can handle working in pairs, but I don't like working in groups.

SK: Why is that?

G: You've seen it. Nobody does the work. We'll be taught for 10 minutes and then the last 5 minutes we'll do the work real quick and you just don't learn as much. If you're in pairs, you know, like me and Jeff, we went to the back of the room, just went aside and did our work without having any other conversation. And it helps.

Gene then indicated that, even though he pays little attention in class, he thinks that mathematics is important, and he sort of likes it more now:

SK: Is there anything else about math learning that you think I should know? Like from a student's perspective. Or about mathematics?

G: I used to hate math. I thought it was just, you know, dumb. Adding and subtracting. And it's really neat after you get into it. I kind of like it. I know I don't pay attention much in class, but I'm sure it will help me.

SK: So you think that what you do—some of what you do in class do you find at least interesting even though you appear not to pay attention?

G: I don't think they'd teach us if it wasn't, you know, if it wouldn't help us. If it wasn't useful they wouldn't teach it.

SK: What about the stuff you had before, like in other math classes? Do you think that will be useful too?

G: Yeah.

Although we had a difficult time trying to induce Gene to talk in much detail about his disengagement, these exchanges are not inconsistent with his observed behaviors in class. Then, approximately one week after this interview, and much to our surprise, Gene's behavior changed dramatically.

During a geometry unit Sandy was guiding a discussion about polygons. As she began writing a table of different polygons and the sums of their interior angles, Gene volunteered his observation that he noticed a pattern related to the increase in degrees of the interior angles each time the number of sides of a polygon is increased by one. Not only did the sum of the degrees of the interior angles increase by 180, but Gene hypothesized that this was somehow related to the fact that 180 degrees is the number of degrees in half a circle. The rest of that class period and the entire next class period were spent discussing what Sandy began referring to as "Gene's theorem."

By the third class, Gene had typed a table to represent his theorem, which curiously he referred to as "Ms. Bethell's conjecture supported by Miss Smith and Mrs. Jones" (See Table 1).

His enthusiasm did not diminish quickly. He insisted that Sandy call university mathematicians to investigate whether this pattern really is related to circles or semicircles. Sandy did talk to a few mathematics and mathematics education professors, and one came to class with Steve to hear Gene present his theorem and to discuss the mathematics related to it. Steve conducted a second interview with Gene to try to shed light on this newly demonstrated engagement with mathematical ideas:

SK: And what I want to ask you is—I interviewed you about a week and a half ago.

G: Yeah.

SK: A week and a half or two weeks ago and we talked about how you didn't try real hard. Then all of a sudden last week I came in and you were extremely engaged and you were showing—what was that activity where—Oh, yeah. If people didn't understand the rounding off you were showing them how to make certain triangles.

G: That's when I changed the numbers so people would be quiet.

SK: All right. How do you explain that all of a sudden you were more interested and more engaged? If there is an explanation?

G: It's just—I didn't think that, I didn't like it before because it just wasn't interesting. And then I found something that puzzled me and I wanted to keep at it.

SK: And what puzzled you was the—

G: Every time you add an angle or a side you add a 180 degrees. You go from a triangle to a quadrilateral you add 180 degrees.

SK: Is that something that you just thought of yourself or was it—

G: Well, it was on one of our—see we were doing polygons and all that. And we had those funny sides for each one of them. And then we had to add up the angles. And she went around the class and they went . . . up like about 180 degrees each time. And I just wanted to see if there was a pattern. . . .

SK: So for a triangle it is 180, then a quadrilateral was 360, then a pentagon is 180 more than that. 540.

G: 540. Then 720.

We do not have enough evidence to support an assertion that satisfactorily explains this change in behavior. Still, we can say that the classroom environment was supportive enough for Gene to reason publicly about his conjecture and flexible enough to allow the class to address it. We also hypothesize that Gene's inclination to "look for a pattern" was not independent of the classroom environment. It was an environment that promoted this type of inquiry and learning, even though Gene was until now extremely reluctant to participate. And it was an environment that was safe enough for Gene to pursue his hypothesis enthusiastically.

Conclusion

We attempted to demonstrate that notwithstanding significant obstacles, it is possible for a high school mathematics teacher to construct a learning environment that promotes the type of student learning described in the *Standards* (NCTM, 1989, 1991) documents. Meaningful student learning will probably not be certain, obvious, or predictable, and teachers will need to be flexible enough to promote it, and creative enough to assess it. Although changing one's practice will likely be difficult and uncertain, it is also rewarding.

It will not be an easy or quick process to move from a teaching culture characterized by isolation to one that incorporates steady support and collaboration. Even this school environment, which is probably farther along than most in moving in this direction, did not provide this teacher the kind of support that she would have liked. Nevertheless, she was able to promote the kind of mathematical learning that is rewarding for the entire community of learners who comprise her classroom.

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