Expanding the Equation: Learning Mathematics Through Teaching in New Ways

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Over the past decade, various reports have identified serious deficiencies in mathematics education in the United States (Mathematical Sciences Education Board, 1989 &1990; McKnight, et al., 1987, National Commission on Excellence in Education, 1983). The National Council of Teachers of Mathematics (NCTM) responded to calls for reform with the publication of the *Curriculum and Evaluation Standards* (1989) and the *Professional Standards for Teaching Mathematics* (1991). In these documents the NCTM presents a vision of mathematics education grounded in three areas: cognitive psychology, philosophy of mathematics, and how mathematicians do mathematics. In the classrooms the Standards describe, teachers are:

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- Selecting mathematical tasks to engage students' interests and intellect;
- Providing opportunities to deepen their understanding of the mathematics being studied and its applications;
- Orchestrating classroom discourse in ways that promote the investigation and growth of mathematical ideas;
- Using, and helping students use, technology and other tools to pursue mathematical investigations;
- Seeking, and helping students seek, connections to previous and developing knowledge;
- Guiding individual, small group, and whole-class work (NCTM, 1991, p.1).

Many teacher educators have argued that elementary teachers aspiring to meet such standards would require subject matter knowledge that differs as much in kind as in degree from that which most now appear to possess (Ball, 1992; Ball & Wilson, 1990; McDiarmid, 1992; Shulman, 1986). They assert that teachers' knowledge is suspect in three areas: knowledge of the content of mathematics, knowledge about the nature of mathematics, and attitude toward mathematics.

First, some mathematics educators question what prospective teachers learn of the content of mathematics as students in elementary and secondary schools. As Ball (1990a) points out, their understanding of mathematics is often procedural and fragmented. The prospective teachers that Ball studied were able to solve problems by following standard algorithms. However, they were unable to explain meaningfully the mathematical reasoning which lay behind those algorithms. They did not seem able to connect in a coherent way the various bits of mathematical knowledge they had accumulated. Evidence from the National Center for Research on Teacher Education's (1991) Teacher Education and Learning to Teach study supports Ball's contention. For example, researchers found that most prospective teachers were able to solve the following problem: What is 1¾ divided by ½? However, very few could devise a context in which such a problem would make sense. Some proposed problems which involved dividing 1¾ by 2. Others, while recognizing that sharing pizza between two people did not represent the problem accurately, were unable to create one that was.

Not only do teachers need connected, sensible understandings of the content of mathematics, they also need to understand the *nature* of mathematics. Ball and McDiarmid (1990) argue that knowledge about mathematics includes knowing: (1) distinctions such as convention versus logical construction, (2) relationships among mathematical ideas, and, (3) the nature of the fundamental activities of mathematics—looking for patterns, making conjectures, justifying claims, etc. (pp. 9, 10).

McDiarmid (1992) presents three reasons for increasing the subject matter knowledge of teachers. First, teachers need to know mathematics in order "to help their learners develop similar understandings." Second, the teacher's stance toward the subject matter communicates a view of the nature of the discipline to her students. Finally, McDiarmid argues, there are "critical ties" between subject matter knowledge and pedagogical content knowledge (Shulman, 1986). Shulman invented this phrase to describe the knowledge that enables a teacher to "build bridges between learners from a variety of backgrounds and the subject" (McDiarmid, 1992, p. 9). Ball (1990b) discusses the ways in which knowledge of subject matter enables her to create a variety of representations of negative numbers for third graders.

Mathematics educators also worry about teachers' attitudes towards mathematics. The literature on math anxiety, while generally not scientific, is extensive. Cross-cultural studies (Stigler and Baranes, 1988) suggests that American, more than Japanese or Chinese, adults, attribute success (or failure) in learning math to "ability" rather than effort or opportunity to learn. The belief that the ability to think mathematically is predetermined can influence teachers' interpretations of their own math history. It can also influence pedagogy: If some people just can't do math, then teachers cannot expect all students to understand what they teach.

So, how can teachers learn what they need to know of mathematics in order to teach in new ways? An obvious suggestion is that they return to universities and take math courses. However, most teachers avoid such immersions. We seldom see elementary teachers (either in- or preservice) in college calculus courses. And McDiarmid (1992) has argued that, if they did take college math courses, the kinds of experiences they would encounter would not promote the kinds of knowledge of or about mathematics that math educators advocate. Moreover, such courses, with their pre-constructed syllabi and emphasis on coverage, are unlikely to alleviate math anxiety. Thus teacher educators have looked for alternative approaches. Some of these involve giving teachers opportunities to be learners in very different settings—settings similar to those the NCTM Standards advocate for K-12 classrooms.

Educators taking these alternate approaches often start with the same social constructivist perspectives on learning that drive much of the current effort to reform public schools. Drawing on the work of Vygotsky, various authors (e.g., Harre, 1989; Wertsch, 1985) have argued that learning takes place in social interactions: Students must have opportunities to make public their thinking, thus making it available for criticism and re-formulation. Ball has suggested (1990b) that student conversations that include conjecturing about mathematical problems and ideas help students to develop an understanding of the nature of mathematics. Cobb (1989) has argued that "each child can be viewed as an active organizer of his or her personal mathematical experiences and as a member of a community or group [which continually regenerates] taken-for-granted ways of doing mathematics . . . Children also learn mathematics as they attempt to fit their mathematical actions to the actions of others and thus to contribute to the construction of consensual domains (p. 34)." Wilcox, et al., (1991) extend this reasoning to prospective teachers, urging that if we want teachers to develop the knowledge of mathematics they will need in order to teach in new ways, we need "powerful interventions that challenge and yet are safe situations in which students can take mathematical, emotional, and intellectual risks. Creating a community of learners with shared responsibility for learning holds the promise of providing such an environment" (pp. 1,2). Because few university math classes create such communities, some teacher educators have attempted to deepen and broaden teachers' knowledge of mathematics by alternate approaches (e.g., Shifter & Fosnot, 1993; Duckworth, 1987). Duckworth and her colleagues met with a group of teachers bi-weekly over a period of a year to explore the learning of mathematics among other topics (Duckworth, 1987). The members of the group reported the learning of content, a better understanding of the nature of mathematics, and a changed attitude toward the subject.

The SummerMath Program at Mt. Holyoke College has created the opportunity for in-service teachers to experience learning in the kind of classroom situation that the NCTM Standards asks them to provide for their students. The program gathers groups of practicing teachers who meet for two weeks in the summer to explore mathematics from elementary and secondary curricula. The teachers also interview children to probe their mathematical thinking, design a lesson based on that knowledge, and teach it. They explore the mathematics in groups of three or four. The teachers that Shifter and Fosnot (1993) describe experienced reduced levels of math anxiety in the supportive atmosphere of the groups. This relaxed atmosphere provided them an opportunity to think about mathematics without fear of evaluation. The members of the groups had opportunities to experience success in thinking about and solving mathematical problems.

The results of these interventions seem promising. The authors report that many of the participants have developed more positive attitudes towards mathematics. Furthermore, their understanding of the nature of mathematics changed. Many participants no longer view mathematics as a set of rules to be memorized, which are beyond their understanding. SummerMath teachers like Sherry Sajdak have "developed a new understanding of mathematics" (Shifter & Fosnot, p. 112) while those like Ginny Brown have learned mathematics in their own classrooms, from their students (p. 158). No longer confined to the state of ignorance they resigned themselves to as children, they are expanding their mathematical horizons.

Projects such as SummerMath, while immensely helpful for participants, and possibly for their colleagues, can handle but a tiny fraction of teachers. So, how can the reforms called for by the NCTM be successful? This paper explores, through the experiences of three primary grade teachers, the possibilities that teachers who begin to teach mathematics in new ways may grow significantly in their knowledge of and about mathematics *through their teaching.*

The teachers are members of Investigating Mathematics Teaching (IMT), a project of the National Center for Research on Teacher Learning (NCRTL). This group of seven teachers and three researchers started meeting in the fall of 1991 to explore a multi-media collection of materials documenting teaching and learning of mathematics in two elementary math classes, one of which was taught by Deborah Ball. [1] During that fall, Helen Featherstone, Lauren Pfeiffer, and Stephen Smith structured activities around watching videotapes of Deborah Ball's third grade mathematics class and looking at Ball's journal and those of her students. They also visited the seven participating teachers' classrooms and interviewed each teacher on a regular basis. In January of 1992, the focus of discussions in the meetings began to move toward conversations around individual teachers' practices. The group has continued to meet on a bi-weekly basis during the school year. Helen, Lauren, and Steve continue to visit classrooms and interview teachers.

Helen, Lauren and Steve first began to think about the possibility that teachers who begin to teach math in new ways might learn subject matter *from their students* in February, 1992, as a result of a conversation between Carole Shank and Helen, in which Carole spoke eloquently about changes in the way she saw mathematics (see below). As they analyzed data from the early phase of the study, they began to suspect that other teachers had also made significant changes in their understandings of the math they taught, their perceptions of math and what is involved in doing math, and their perceptions of themselves as learners of mathematics. In the fall of 1992, when they invited teachers in the IMT group to collaborate in looking at IMT data on this issue, Carole, Debi Corbin, and Kathy Beasley were particularly interested. The three cases that follow are the result of a collaborative effort to tell their stories.

Although the other four teachers in the IMT group hail from four different districts, Debi, Kathy, and Carole all teach at the same urban elementary school. Carole and Kathy have been colleagues there since 1983; Debi is a relative newcomer, having been assigned to student teach in Kathy's second grade classroom in 1990 and having then stayed on in the school as a "co-teacher" assigned, as part of Averill's Professional Development School effort, [2] to provide restructured time to a team of four primary grade teachers—a team that includes Kathy and Carole. In the fall of 1991, when Helen, Lauren, and Steve were recruiting teachers for the IMT group, Kathy and Carole had just moved out of the grade level teams in which they had both been teaching since they arrived at Averill in order to follow their second and third grade students for two years. The new structural arrangements and shared interests in new ideas about teaching led the three teachers to spend many school lunch periods talking about teaching and to join the IMT group together. All three were interested in learning more about Deborah Ball's math teaching. But because none of the three felt at all confident about her own knowledge of mathematics, all were also somewhat nervous about the announced focus on a unit from the 1989-1990 school year in which Deborah had introduced her third graders to operations involving negative integers.

CAROLE

> I thought math was very, very individualistic and *dry*. There was a process [algorithm], you had to learn it, and you got through it. And I had trouble memorizing processes often. I wasn't good at math in high school, I wasn't good at math in college, and so I avoided it because I wasn't good at memorizing what to do and how to do it.
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> I never really got what it was about, and never even really realized that even memorization would get me through it. I just kept trying to figure out . . . and it just never made any sense. I knew my facts, my addition, subtraction, multiplication and division facts, and where that didn't work, I avoided it. My husband would measure stuff and he'd ask me "How far do you think it is?" and I'd just say, "I have no idea."

Like many, perhaps most, American adults, Carole has felt inadequate in relation to mathematics for a very long time. This sense of inadequacy started at least by the time she was in high school. It continued through her college years and was well established by the time she became an elementary school teacher. Unlike Debi, who memorized formulae and felt able to do problems that looked like the ones she had practiced, Carole had little confidence even in her memory. Unable either to memorize formulae or to make sense of the material, she felt that she was without tools for dealing with mathematical problems.

Because she felt incompetent mathematically, she kept math out of her life as much as she could. And the example she offers suggests just how far out of her life it was possible to push it.

Over the past year and a half, as conversations about mathematics problems have come to occupy an increasingly prominent place in her math teaching, she has come to a different view about what mathematics is, about what it means to do mathematics. She has also come to *feel* very differently about math. These changes in her perception of what mathematics is and how one does math, and in her feelings about math, have brought about changes in the way she sees math operating in her own life.

Twenty Years of Traditional Math Teaching

For the first twenty-three years of her teaching career, Carole taught third grade in the same elementary school and "teamed"—taught and planned—with the same colleague.

> I pretty much followed the book, followed the curriculum guide. It was pretty much rule oriented, I've come to believe. Once in awhile we'd use manipulatives to show them something, but they weren't really tools that kids manipulated. I wouldn't really refer to it as a tool, it was more like a demonstration. It was more *my* tool, to show them something. It wasn't for them to use. I didn't know how to have them use them. Or even how to watch them to see what they could do.
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> And there was always the routine assignment of problems involved. Problem after problem. And then we got so you don't have to do every problem, you can do every other problem. And that was a big step. And then we got to the

> point where you didn't have to do every page, you could skip some. And those were all big *steps*. It sounds ridiculous, but that's the way it was.

Given her own experiences with math, it is perhaps not surprising that even moving from assigning all problems to assigning only those with even numbers seemed like a major step. In 1989, at the PDS Summer Institute, she saw Deborah Ball teaching math for the first time:

> I guess I started to see something new when we went to our first summer work shop, and [Deborah] Ball was there. And the things that she was saying and doing were unbelievable. . . . She had brought in some kids, she showed videotapes. She gave the kids some problems, and asked them what they were doing and they would explain what they were doing and she would ask them to explain what they were doing. We looked at their journals and we saw those kids in real life.
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> And the things that she was doing were so different from anything I was doing. The conversation, the discourse, the discovery, the accepting of answers, kids listening to each other, the teaching from one child to another. And she was pulling from the conversation things to move to the next step. I was amazed.
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> That was my initiation, my first real experience of something different. Then, there were different times, throughout the three years after that that things happened. And then, somewhere along the line, we found Marilyn Burns, and she seemed to have all the answers. And that gave me an avenue of experimenting with math differently. We started doing some of her activities, but the discourse wasn't there: it was the activities that we were looking at. So the activity became the thing that sort of got at the understanding of what multiplication was. But we were still doing the drill, and all the multiplication facts and all the addition facts, and all the timed tests, and we were still doing all the written stuff. Lots of homework.

During 1990-91 she described her math teaching as a mix of Marilyn Burns activities and drill and practice worksheets. "I did some multiplication stuff from Marilyn Burns, but then I'd slip back into the workbook."

In 1991 several things happened:

- Having joined the team that was setting directions for her school's Professional Development School effort, she was thrust into a series of conversations with several of her colleagues about the role of teachers, women and power, and open communication. She explains:

  > We had spent *hours* and *hours* and *days* learning how to talk to one another as members of this management team who didn't know what it was about and what we were supposed to do and what we really wanted to do and why we were even there.

  These conversations led her to think more about the role she was taking in the school, and about changes she needed to make if she was to grow professionally.

- Because she decided to follow the same group of children for two years, she moved to second grade and stopped planning with the colleague she had teamed with for the previous 23 years.

- She began lunching regularly with Kathy and Debi and talking with them about teaching. Kathy had taken a course over the summer on the NCTM *Standards* and many of these lunch time conversations focused on math teaching. In late September, when Kathy and Debi began to think about joining the IMT group, they suggested that Carole join them and she agreed to do so.

October, 1991

Carole had embarked on an adventure when she moved to a new grade and out of a comfortable and familiar teaching team. She joined the IMT group because she wanted to learn more about the *Standards* and because she had been intrigued by what she had seen of Deborah Ball's teaching two summers before. From the very first IMT meeting, she made connections between what happened in the group meetings and what happened in her classroom. She also took active steps to create a group. Lauren's journal account of our first meeting notes:

> Carole broke the linear format of "reporting" and made it more of a conversation. She asked questions of the other teachers. She spoke to the teachers and not to us!

When other teachers raised questions about the difficulties the third graders in Ball's class were having understanding why 200-190 was not 190, she related the children's difficulty to an observation about her own students: students this age have difficulty understanding 0 as a placeholder.

Just as she carried what she saw in her third grade into the IMT group, she took what she saw in the first IMT meeting back to her classroom. At that first meeting we showed videotape of Ball's third graders discussing number sentences they had generated in response to her request that they "write number sentences that equal 10." Before watching this tape, members of the IMT group did the task themselves. For Carole, writing her own number sentences, examining the number sentences that Ball's students had written in their math notebooks, listening to their discussion of these sentences, bringing the task to her own third graders, and observing what they did and said became both a pivotal event and a metaphor for changes in her own view of mathematics that grew out of observing and listening in a new way to her students as they grappled with math.

Here's what Carole wrote in her notebook that first night. The first column responds to Helen's request that everyone write number sentences equal to 10; the second that they write number sentences equal to ten that they thought a third grader might write:

Write number sentences = to ten

3+2+5	6+4
6+3+1	5+5
4+4+2	2+8
3+1+2+2+1+1	1+9
11-1	9+1
	4+6
	3+7
	7+3
	11-1

All of Carole's number sentences except "11-1" involved additive combinations of integers between 1 and 9. She was, she remembers, surprised at the much wider variety of sentences that we found in the notebooks of Ball's students when we looked at them later that evening—equations like 100 ÷ 10=10 and 200-190=10.

A few days later she gave her own third graders the same task; like Ball's students, they ranged adventurously across the numeric territory they knew. A year and a half later she remembered the scene this way:

> When I looked at that problem, "Write all the ways you can write 10," I thought, "Hey, I can do this, I can do this, I can do this, I can do this." . . . Then when my kids did it . . . all the different ways to look at ten, I thought, "Wow, I didn't realize ten was out there all those different ways."
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> But that happens a lot.
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> And I took my first directions from my kids, I think. And then it was encouraged, I think, by conversations with colleagues, in [the IMT group], with Kathy, with Debi. They were getting excited about it, too. I wanted to know more: I wanted to figure out what else the kids knew.
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> It surprised me that those kids would figure that out. . . . When Jason [3] came out with "It's 200-190," I thought, "Look at what he's doing! And I was just copying something that [Deborah Ball] did."
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> Then when my kids did it, and when [Deborah Ball]'s kids did it I thought "Wow, there are all these different ways to look at 10?"

Three months after giving her students this task, reflecting in a conversation with Helen on the changes she had made in her math teaching, Carole referred to this experience and the way it seemed to represent for her the way a world that had seemed to be tightly sealed was beginning to crack open for her:

> What has been a real awakening for me, I think, as much as anything, is the relationships in number. I really never saw much relationship before. I mean, addition's addition and carrying was related to addition and borrowing was related to subtraction. But now the world of number is really exciting for me. When I see the combinations of numbers that [the students] get with the mini-computer or the combinations that they got with the problem that [Deborah Ball] gave, "What's 10?" and some of the things they are coming up with. And I

> always thought 10 was 6+4 and that there were certain facts. . . . But it's a huge world of 10 out there, it's a whole world of all different numbers and I always looked at it as a very narrow thing. . . .
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> And that is really growing for me this year. It's exciting. It really is.

"I wonder with my limited math background if I can do it?"

Even though she was intrigued by the number sentences her students wrote during this lesson, Carole felt very uncomfortable about her own knowledge of mathematics. On October 15, after attending two meetings of the IMT group and spending some time reading Ball's journal she wrote:

> Reading [Ball's] journal is helping me to see the process she goes through deciding what to teach, how to follow one piece of the curriculum to the next. Making these connections is essential. I wonder with my limited math background if I can do it.

Subject matter knowledge seemed critical to the kind of teaching she saw Ball doing in the two videotapes we had watched. Indeed, even following Ball's thinking seemed a bit of a reach to her. Because she was also feeling very short on time, she wondered whether it made sense to continue in the group. She concluded her October 15 journal entry with this question:

> I'm finding it difficult to have time to think about my own curriculum and what their math understandings are, let alone trying to follow [Ball's] thinking. Maybe [the IMT group] isn't the place for me right now?

As soon as she read this journal, Helen called Carole to talk and to say that she hoped Carole wouldn't drop out of the group. Carole reported immediately that she was feeling much better. She was very excited about that day's math discussion:

> We were talking about ways to equal 12. One little boy had written "100-100+12=12." Some of the other kids were confused. Another little boy explained it, saying, "It's like 1-1=0.' and 0+12=12" The other child understood!
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> I was so excited because I had taken the time to talk and listen to them! . . . I haven't done this before, but I'm just so excited!

Once more it was her own students—their ideas, their success at explaining their ideas to others and to her—that had made the difference. She was excited both by what she was learning about her students—"I had taken the time to ask them, and I'm hearing their ideas"—and about the mathematics that was surfacing in the room.

Carole was still troubled, however, about the chasm she saw between her own knowledge of mathematics and Deborah Ball's—"I don't know the math that she does"—and for the same reason: "Where do we go tomorrow?"

And the IMT meetings were not always easy either. Reflecting back on the year during the following summer she recalled that she had often had trouble following what Ball's third graders were saying in the videotaped discussions that the IMT group watched:

> I really had trouble. . . . I would get halfway through the conversation and think, "what are they talking about?" I had completely missed the whole thing. But I'm better. It's not easy being a listener. So I don't bring in my interpretations but just listen . . . for what they are saying, you know, and try to interpret what they are saying. . . . There is a fine line there, in between how I construe it so that it makes sense to me but so that I could communicate what they're actually saying.

Hearing and celebrating the students' ideas

An important theme in what Carole wrote and said during the fall of 1991 was the pleasure she got from hearing the children's ideas in math. At our first meeting she had written that a central concern for her was to get the students to listen to one another: "They all want to share their solutions but they don't want to listen." As the weeks went by she referred to this concern from time to time, but what was most dramatically clear was the pleasure she was getting from listening to them and learning about their ideas.

In January she talked about the central role pleasure and satisfaction needed to play in teaching and learning. And she connected her own satisfactions to what she was learning about what her students knew:

> I'm amazed at how much knowledge kids really have. I'm always amazed. I don't think a class goes by that I'm not amazed. They know a lot. There are a lot of them who know a lot about a lot of things. And you don't discover that unless you let them share and let them talk.
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> It goes back to being open to change. I keep thinking, I'm struggling so hard right now, but I'm not preparing for something: this is life right now. This is part of it. If I can't get what I want out of it right now, I've got to make some changes, because I'm not preparing for anything. I've got to appreciate what this is. For some reason that really hits for me. Because I hear so many people say, "I'm preparing for retirement," "I'm preparing for this." But you aren't going to do it then either. THIS IS LIFE. You aren't preparing to graduate, you aren't preparing for anything. This is it. What can you get out of this right now, and what can we do with this?

This emphasis on the importance of living life in the present suggests another connection: In October she saw her lack of subject matter knowledge as an impediment to planning, to using the children's comments and insights as a basis for moving the curriculum forward. In October she seemed to be saying, as she reported on the excitement of good discussions, that she felt competent to orchestrate the discussions that were occurring *in the present* but worried about long-range planning. She saw Ball's knowledge of math playing a central role in her planning, in the decisions about "how to move from one piece of curriculum to the next." In January, as she affirmed the importance of the present, perhaps she was also reminding herself of the importance of *what she was able to do.*

Summer Reflections

Looking back over the year in the summer of 1992, in a conversation/ interview with Steve, Carole touched on some new changes in her thinking about math and the learning of math:

> What I thought was understanding is no longer anywhere near where I see my students, what they talk about, how they talk about math. . . . I've really developed the confidence that they can figure this out. Where before I never really thought about it, I guess. ..I just thought math is just [writing down a problem] and spitting it back out on the paper. You know, it was just kind of pushed through this hole. I don't talk about it very eloquently, but it's just so different: Before it was just pushing out problems and pushing out . . . the right answer. And it's not there at all anymore. And it's outstanding, I marvel at that, I really do.
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> I don't know how to do it well, . . . but it just feels right.

When Steve asked about whether she saw changes in her own view of mathematics, Carole answered, "I'm not afraid of it because I can figure it out, too." She was still unsure, she continued, about negative numbers, "and I'm still not sure where the next step is when my kids are struggling with a concept . . . but I feel I have people I can go to for help."

Six months later, when we all wrote individually about changes in the way we viewed math, Carole described her thinking this way:

> Math always seemed a pretty black and white subject before. You followed a procedure, you followed a process, you got an answer. It was individualistic, not shared except with the teacher or the checker or whoever was involved with it, and you moved on.. Not even much relevance to the real world. Except subtraction and my checkbook.
>
> Since joining this group, I guess, and learning about the NCTM Standards and [Ball's] tapes and having discourse about what's important and how to do it this way or that way, or whatever, math no longer is an isolated thing. It's communication, it's a discovery, it's an adventure. All answers are different and varied. It's about how we think and about how numbers work and about how it all works together. Math now has life, it has many questions and lots of answers and a wonderful way of manipulating all different numbers. When I think of the teaching, like, of place value in my classroom and watch how hard kids are working to figure out what tens, hundreds, and ones mean and what does it all have to do with addition and subtraction and multiplication, and see the emerging discourse and the problem solving that is going on, I find it just really really exciting.
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> Math has become very obvious in my life as far as in the outside world. I can't get specific about those, I'd have to think more about those, but I see it much more as part of my life. And I feel like I've only just begun. I'm nowhere near the end; I think it is just an ongoing process that I've started and I'm really excited about.

When Helen asked her why she thought these changes had occurred she answered:

> Partly because it's open. It's not just one procedure that I had to memorize but it's ways of discovering how different numbers work. That's the release for me, that's the thing that has opened it up as much as anything.
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> And trying to see how other people [children in the class] are thinking. . . . When I look at some of their solutions, I think, "That can't work. No, wait a minute, that does work!" It's shocking sometimes the way things that they do that are right there in front of me that I would never have picked out in a hundred years, they saw it that way. It's a whole different way of thinking about it.
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> I have faith that my kids are going to come up with the answer, or with some way of thinking about this. . . .[In the past] I was looking for an answer—the number at the bottom. I may not even have known the answer myself. I looked in the teacher's guide and checked off the answers—not having any concept that there was anything beyond the rote process.
>
> So it's very different for me.

Where is Carole today, in relation to Subject Matter Knowledge?

> It's still an issue. I'm taking more cues from the kids. But I try to know where I want to go next.

From the start Carole has seen subject matter knowledge as intimately connected to issues of planning and knowing where to go next, seems like a bit of a resolution. But clearly she does not feel that she knows all she needs to know about mathematics in order to teach math well.

Reflecting on Carole's Story

Carole had arrived at a dead end in relation to the learning of math well before she even entered college. She had a view of math which made it highly unlikely that she would ever learn anything she did not already know. Math presented a smooth closed surface to her: it was "black and white," it was solitary, and it was irrelevant to her life. When something that looked like math appeared in the doorway—when, for example, her husband asked her to estimate a distance—she shut the door, declaring firmly, "I don't know." Because she and math had agreed to live separate lives, nothing much changed in their relationship. All this has now changed; math beckons to her both in the classroom and outside of it. Her students' mathematical insights intrigue her; she tries to follow their thinking and sees the world of number expanding. Her story recalls the moment in the Wizard of Oz in which Dorothy opens the door to her black and white house and realizes that she is not in Kansas anymore.

KATHY

Kathy has always taken her own learning seriously. So seriously, indeed, that she changed her major from elementary education to French in her sophomore year of college because she felt that she wasn't learning much of value in her teacher education courses.

After graduating from college in the late 1960s she moved to northern Florida where she spent one year as a VISTA Volunteer and another teaching high school French. She then moved to Michigan with her husband where she left teaching to begin raising a family. After her second child was born, Kathy obtained her elementary education certification; she began teaching at Averill elementary school in Lansing Michigan in the mid-1980s.

During her first couple of years at Averill, Kathy taught second and fourth grades using many of the traditional methods she had learned. Then, however, she and her team teaching colleague became interested in new approaches to teaching reading. Over the course of the next few years they abandoned basal readers and ability-based reading groups, to move toward a "whole language" approach to literacy. Although these changes were rather unsettling at first, Kathy and her colleague were reassured and very excited by their students' responses to the whole language innovations. As they became more proficient using the whole language approach, they begin to explore the idea of whether it would be possible to make changes in their mathematics teaching that paralleled the changes they had made in the teaching of the language arts.

In the summer of 1988 Kathy became very interested in the ideas that were presented in a "Math Their Way" workshop, sponsored by the school district. Following the workshop she began to try new approaches to teaching math— for example, she found the books of Marilyn Burns very helpful— and to seek out others who were making changes in their practice. This led Kathy and her colleague to attend, in the summer of 1989, a Michigan State University (MSU) Professional Development School (PDS) summer institute that included a workshop on mathematics.

They saw video tapes of Deborah Ball and Magdalene Lampert teaching math and watched Deborah work with several of the eight- and nine-year-olds she had taught the previous year. Much intrigued, Kathy began to further experiment with new approaches to teaching math.

In the fall of 1990 Kathy and MSU teacher educator Sharon Feiman-Nemser initiated a PDS project that involved long, searching weekly conversations about teaching. These conversations encouraged Kathy to experiment in a reflective way in her approach to teaching. Kathy says, "She helped me understand the joy and intellectual work that is what teaching is all about." Sharon put Kathy in touch with people at Michigan State University who were interested in alternative approaches to teaching mathematics.

In the spring of 1991, Kathy joined a study group that Deborah Ball and Janine Remillard had organized for student teachers they had taught in a math methods class. The following summer she took a graduate class on the NCTM Standards. In the fall of 1991 Helen approached her about joining the IMT group. Kathy hesitated: she had a number of other outside commitments; in addition, she recalls, "I thought I was going to be expected to be more knowledgeable about math than I knew I was." She warned Helen of this worry on the phone, saying, "Helen, I have to tell you that, when you say negative numbers, it makes me feel very anxious." However, she was so much attracted to the idea of learning more about Ball's teaching methods that she decided to take the plunge.

Fall, 1991: Feelings about Math and Math Teaching

The first paragraph of the journal that Kathy kept for the IMT group captures some of her feelings about math and math teaching:

> Did I say I hated math at our last class? I feel badly about that. I don't really hate it anymore. Maybe I never did. It is far more accurate to say I fear math. But it feels more powerful to say I hate it. I guess that's why I said that. I know I would be offended if someone said that they hated literature. What I really want to do is understand math so that I won't be tense and worried about it. Mostly, I never want my students to fear or hate math. They all seem to love math. And really I do like teaching it. Teaching math has helped me understand math.

A few days later Steve visited Kathy's third grade class and watched her teach a math lesson inspired by Marilyn Burns. As she had noted in her journal, students seemed to be thoroughly enjoying themselves and their task.

On the board Kathy had written lists of items costing $3, $4, and $5 in preparation for an imaginary shopping trip. She told students that they can "spend" $10, that they should decide what they want to buy and why, figure their totals, and explain how they arrived at the figure they did. They worked on this task alone or in small groups, devising a variety of methods for keeping track of their purchases. After the class reconvened, groups shared lists, methods of computation, and totals. Kathy concluded the class by asking the students to look for patterns—one noted that "All the numbers in the tens place are 1"—and telling them that the next day they would talk about what the totals would add up to.

I'm Jealous. . . .

Kathy's first journal entry also highlighted two issues that compelled her interest across the next year. The first was listening:

> I am also trying to think about children listening and learning from one another. I want this to happen in my classroom. I've been thinking a lot about listening. My children listen to each other best in the morning during sharing time. Each child shares one thing, anything they want. They can't talk when someone else is talking and they really observe this. I am not in charge. A different child is each day. Michelle is in charge of keeping track of who is in charge and making sure everyone gets a chance. Anyway, they listen to each other: They talk to each other. It seems to me that the reason Deborah's children listen to each other is that the questions they are discussing are theirs. This is a really clear example of responsive curriculum I think. How is this different than what I do in math?

The second was the language Ball used to describe her third graders mathematical thinking and the knowledge of mathematics that stood behind this vocabulary:

> She had labels for children's thinking—like "decomposition of numbers." I was interested in her note about the "compare meaning" of subtraction. "I know from experience that it is the harder meaning for children to understand. Using pictures and comparing the amounts with manipulatives seemed to be the only way we could make sense of that meaning." I've reread her entry a couple of times and I'm not sure what she is saying. What does she mean when she writes "This problem is interesting in part because their ability to reason mathematically depends on their understanding of the compare meaning of subtraction"?

At the group's next meeting she spoke of feeling "jealous" of Ball's command of a vocabulary for making these mathematical distinctions. Although no one else in the group took up this topic, Kathy continued to show considerable interest in following Ball's thinking when she ventured more deeply into mathematics and when she used unfamiliar mathematical vocabulary. In early November, for example, when we examined a chart that Ball had made to compare the advantages and disadvantages of various representations for extending her students' understanding of negative integers, Kathy asked more questions than anyone else. Even though she remained less than confident in the numeric territory below zero, she continued to try to make sense of operations with negative numbers and to use her own efforts at sense making to assess what she saw in videotapes of Ball's third grade.

The Attack of the Killer Elevens

Although still uncomfortable in the realm of negative numbers, Kathy was making changes in the way she taught mathematics. In late January, in order to explain the character of these changes, she described a recent math lesson to Helen. Kathy had begun, she explained, by asking her students to compare 30 and 19.

> But I didn't stop there. I said "use your minicomputer to figure this out, and then explain how you figured it in your journal. And then, if you finish, here are some other problems." I knew I had to have something else for them to do while some of the others finished the problem. So I gave them a whole series of problems. And as I got to creating them I thought, "Oh, I'll do a pattern and they'll all come out the same and I'll see what they do with it."

After the students had worked on these problems independently for a while they reconvened and looked at the first problem together. They agreed without much difficulty that the answer was 11.

> So, we finished with it, and everyone was feeling pretty good about it, except that one of my students, Lisa, tried to talk about how 30 take away 19 and 50 take away 39, which were the only two problems she had done, were the same.

In trying to articulate this Lisa came to the board and wrote

$$\begin{array}{r} 50 \\ \underline{-39} \\ 11 \end{array} \qquad \begin{array}{r} 30 \\ \underline{-19} \\ 11 \end{array}$$

> And she's seen a pattern! . . . Which I thought was interesting. So, I decided to pursue that with the kids.

The next day, Kathy gave her students a series of problems like this:

30-19=
50-39=
60-49=
90-79=
40-29=
80-69=

asking the third graders to work individually on the problems and to look for patterns. Several students responded immediately that they knew that all the answers were 11. Kathy said that this was fine; they could just write down 11 and then start looking for patterns.

When her class reconvened, students talked excitedly about the patterns they saw: they noticed that they were adding first 10, then 20, etc. to both the top and the bottom number, that all the top numbers ended in zero and the bottom numbers ended in 9, etc. Then they got interested in what would happen if they added some number that was not a multiple of 10 to the top and the bottom numbers. They tried 7 and were surprised to see that the difference was still 11.

"Then," Kathy reported, "someone said, 'Well, this is the attack of the killer elevens, we have to get rid of these elevens!' And I said, 'Well, how would you get rid of the elevens? What would you add?' The third graders became much engaged with this question. They tried adding several different numbers with, of course, no luck. Then Nathaniel called out excitedly, 'Eleven! Let's try 11.'" There was a murmur of excited approval. As a group, the third graders appeared to be convinced that if they added 11 to both the 19 and the 30, the difference could not continue to be 11. They were again astonished by the results of their arithmetic:

$$\begin{array}{r} 19 \\ \underline{+11} \\ 30 \end{array} \qquad \begin{array}{r} 30 \\ \underline{+11} \\ 41 \end{array} \qquad \begin{array}{r} 41 \\ \underline{-30} \\ 11 \end{array}$$

Finally Cindy got Kathy's attention: "Mrs. Beasley, I've had my hand up for an *hour* and you *never* call on me." After making her way to the front of the room, she turned triumphantly toward her classmates: "You're all wrong. 60-49 isn't 11: It's 29!! See [writing it on the board]: 0 take away 9, you can't do it so you write 9. 6 take away 4 is 2. 29! So, if you add 30, you get 29, not 11!"

The third graders stared at the numbers on the board, and then many agreed! Some didn't. All this despite the fact that Kathy had worked extensively with regrouping only a few months earlier and, with most of the students (she had taught second grade the previous year), the year before as well. Gregory disagreed. He said that you *could* take 9 from 0, and that, when you did this, you got -9.

After telling this story, Kathy returned to the issue that had been puzzling the third graders before Cindy took the floor:

> I started wrestling with, "How am I going to help children understand that the space, the amount between those two numbers, remains constant as long as you're adding the same amount to each of them?" I don't even know if I understand this really clearly. So I don't even know what to do with that, Helen. I think that's the whole issue. I'm really in a bind here.

The students were clearly exploring unfamiliar patterns and asking challenging questions of the numbers. Kathy was very excited about their extended exploration; the lesson felt quite different from the one Steve observed in early October.

Discussing this lesson with Helen four months later, Kathy identified *the way in which she was listening to the students* as the key difference between this lesson and the ones she had been teaching earlier. A comparison of the two lessons helps us see what she meant. In the October lesson Kathy clearly listened with interest to the children's lists and observations. However, there is far more to listen to and for in the January lesson.

Revisiting her comment in January, 1993, Kathy explained, "It's not that I didn't listen before, it's that I didn't let them say anything." Continuing this line of reflection she went on to say that she thought that she had been so focused on correct answers that there wasn't that much interesting for the children to say. Then she stopped herself in mid-sentence to revise: "You know, I probably *didn't* listen. The whole structure didn't allow for the children to say anything, so there just wasn't anything to listen *to*." She shook her head disbelievingly, "What a weird way to teach!"

In January, the children were giving the lesson new direction all the time. They were posing mathematical questions that had not been explicitly on their teacher's agenda when she put the problem on the board. As Kathy pointed out, it is one thing to listen for expected answers and something quite different to listen for, and to, unexpected questions.

This would seem to be the "responsive curriculum" that Kathy had seen in Ball's videotape in October. At that time she conjectured that Ball's students attended because the questions came from them. She was attracted by what she saw and because she believed that if her students' were pursuing their own questions, they too would listen more closely to each other and learn more math. Her "killer eleven" story suggests that she had achieved, at least in this series of lessons, what she set out to do: Her third graders were pursuing answers to their own questions; they seemed to be listening carefully and thinking about what they see and hear. Their teacher was equally excited and attentive.

Near the end of this January conversation, Kathy suggested to Helen that she would like to follow up on Gregory's observation—"You *can* take 9 from 0"—by teaching her third graders about negative numbers, "If I could have someone come in my classroom to help me do it." Helen was dumbfounded: all fall Kathy had declared her discomfort with the idea of below zero numbers; only two months earlier she had declared in class, "It's hard for me to imagine pursuing *anything* to do with negative numbers." Helen took the proposal as evidence that Kathy was as strongly committed to expanding her understanding of mathematics as to altering her pedagogy. Intrigued by the possibilities for learning more about the challenges of this kind of teaching, Helen agreed, after several more conversations, to co-plan a unit with Kathy and spend two to three math periods a week in her classroom.

Teaching about Integers

Over the course of what turned out to be five weeks, Kathy and Helen worked together with Kathy's third graders. They revisited regrouping because many of the children seemed confused about when to regroup when doing subtraction. And, using a thermometer, they explored addition and subtraction with negative numbers. While children worked on problems alone or with others, Helen and Kathy circulated, asking questions and listening to ideas. Kathy orchestrated full class discussions; Helen watched enthralled and made occasional suggestions during class; in the evenings they debriefed extensively and planned next steps together. (See Beasley and Featherstone, in preparation.)

Kathy and Helen were exhilarated by the work, by the children's delight in their own discoveries, and by the richness and diversity of the theories the third graders generated as they explored the thermometer and wrote number sentences that "ended below zero." In addition, Kathy herself was learning to navigate this new numeric territory. On February 23 she wrote in her journal:

> [W]ith this negative number stuff I am learning how to think about it right along with the kids. I was very excited when I understood the strategy Janine and Violet and Jonathon had all been talking about and that Cindy posed the conjecture for: "If you have a problem that is like Violet's (11-9) and the answer is above zero, if you switch it around (9-11) you'll have an answer below zero." When I realized Thursday night how well that works and just felt now I can "do" negative numbers I decided I wanted everyone to understand that. (2-23)

A week later she described to Helen a full class discussion of students' efforts to start with a negative number and write a number sentence equal to zero.

> They all gave examples of how to get to zero. . . . They said it was really easy. Then Noah and Justin gave theirs: "-10+-10=0." I wasn't sure whether it was right or not.

Another student disagreed with Noah and Jeff's formulation, pointing out that [someone else] had shown on the thermometer that -10+10=0 and that, if this were true, -10+-10 could not also be equal to zero. As she listened to the discussion, Kathy saw that -10+-10 would have to be -20. She asked Noah and Justin to ponder the following question: "What if it were -10 degrees in Anchorage and the temperature fell another 10 degrees?"

A year later, writing to Helen about what she learned as she taught the unit, Kathy recalled that, before teaching it, "I had absolutely no confidence in my understanding of negative numbers." As evidence she pointed to a slip she had made in formulating the problem with which she and Helen had planned to launch the unit. [4]

> I think that first problem is evidence of how little I understood [when we started]. What did I learn? I think I learned that I could do really hard math by teaching it. The fact that I understood negative numbers, how to add and subtract them, was very helpful to me. To this day I know that if I stop and think, "draw a thermometer," I can always understand how to add and subtract negative numbers. I think I do have a mental block when it comes to this, but I know it can't really block me anymore, I know I *can* understand this. I feel like I should say I do understand negative numbers, but I honestly don't believe I do enough yet, I can say I *can* understand them and that for me is monumental.

> Yes the children's explorations helped me. It was their thinking that taught me. They could get up there and say out loud my misunderstandings. As soon as they said them I would understand. They unpacked the concepts, the thinking, the parts of negative numbers that I needed and they needed in order to understand. I remember every time someone put forth a wrong answer and explanation I would have to think and test the idea, I didn't just know it was wrong.

Teaching Fractions

After integers, Kathy moved on to fractions; this unit proved unexpectedly difficult. At the end of the first week, she wrote in her journal:

> I am feeling bewildered by math. This way of teaching is difficult. This is the first time I have done this on my own. By that I mean that working with Helen I started really understanding this way of teaching. As long as I had Helen to consult with I was doing great. (I didn't realize how much she was helping me. Not that I didn't appreciate and deeply value her presence and being able to talk to her) I just didn't realize how lost I would feel without that support.

Helen was puzzled by Kathy's assertion that she was teaching differently from the way she had taught before their five weeks of collaborative work: In the lesson involving the attack of the killer eleven she had been listening carefully to children and allowing their questions and conjectures to guide the direction of the conversation; what was different now?

Her visits to the classroom and Kathy's journal descriptions of some of these classes suggested an answer that connected directly to the division of labor Kathy and Helen had established when they worked together: During those lessons Kathy had orchestrated all full-class discussions; Helen, who was thus freed from the responsibility for managing the minute-by-minute interactions of a class of eight-year-olds, attended carefully to the mathematical ideas and theories that children were sailing into and around. As a natural consequence of this division of labor, Helen had taken charge of suggesting a task for the journal writing with which Kathy and her third graders usually closed a class discussion.

Before Helen and Kathy worked together, Kathy's journal assignments followed no set pattern. Some focused on a piece of mathematics: After the "killer eleven" lesson, for example, the children wrote about whether and why they thought 30-19 was equal to 11 or 29. Others were quite general: After a discussion that centered on regrouping, students addressed the question, "What did you learn in math today?" During her time in the third grade, Helen tried to capture some important mathematical disagreement that had been embedded in the preceding conversation. Circulating around the room, reading over students' shoulders and talking to them about what they were writing helped Kathy and Helen to learn more about children's conceptions and misconceptions and to push their thinking. Not wanting to lose this piece, Kathy added Helen's "job" to her own: From the first day of the fractions unit all her journal assignments required the third graders to "do math." An excerpt from a journal entry that Helen made after visiting the class on April 9 suggests both how hard Kathy was listening for and to the children's mathematical ideas and how complex was the task she had set herself. The students had been working on a problem involving dividing 10 brownies among 4 people and had done some very nice reasoning both about the answer and about how they might express that answer:

> [J]ust as it was getting to be time to break for snack, Jonathon, who appeared to have impressed everyone in the class with his command of the division, said he wanted to ask the class a question: "Do you think that 10 divided by 4 is the same as 4 divided by 10?" Some one—or maybe a few people—said no. "Yes," said Jonathon with great authority, "it is. My friend told me. No matter which way they write it, the number of cookies is always the big number, the number of people is always the small number. You always divide the small number into the big number." I think he wanted us to write this down as a conjecture. I realized that we hadn't had any *wrong* conjectures before—perhaps because we had played a role in encouraging kids to formalize their promising ideas into conjectures. I'm not sure why. Anyway, I was wishing that this wasn't happening: I did not want to end the class by writing up a wrong conjecture. . . . He or someone else said something else about how you couldn't move the numbers around in subtraction, but you could in addition and division. His tone of authority was impressive. I did not know how we should respond. . . . I felt that

> he had brought in an authority from outside the classroom, and that the information of this authority would be accepted because the authority was not there to debate with. I looked over at Kathy, and instead of looking as puzzled and worried as *I* felt, she was writing notes in a notebook. When he was done she stood up (looking equally authoritative) and told the kids that she had a journal assignment for them: They were to get their snacks and then they were to think about these three ideas and write down which ones they agreed with, which they disagreed with, and why. The ideas were: (1) that you can move the numbers around in addition, (2) that you can't move them around in subtraction, and (3) that you can move them around in division and that you always divide the small number into the big number. Students adjourned eagerly to their seats and began to write. Kathy wrote the conjecture up in orange [we had instituted the convention of writing conjectures in orange when they were formulated and students were working to "prove" or "disprove" them; we recopied them in blue when everyone was convinced that they were true] as Jonathon insisted. As I walked around looking at notebooks and talking to children I saw that just about everyone was disagreeing that you could move the numbers around in division, and agreeing with the other two assertions. I thought that the assignment was wonderful: the kids got refueled, they saw that it was up to them to really *think* about this idea, and they did. Whoopee. I just lacked faith, I guess. I thought, at that moment, if we were doing a musical about the work Kathy and I did together (a different way to tell our story), this would be the culminating moment: I am completely superfluous.

Kathy was clearly listening to what students were saying for issues of mathematical substance. Because she was *both* orchestrating the discourse moment-to-moment now *and* creating journal assignments that would begin with what children were saying and use it to push their thinking as they worked more individually, she had to think constantly about the mathematical issues that the discussion was raising, and to decide which ones were important enough to pursue. The notebook in which she had begun recording the representations students used and what they said during the general discussion helped her to keep track of the contributions of individuals; equally important, it was also a tool that allowed her to tease out mathematical issues both for herself and for the children. She explains:

> I had to engage in the thinking, the mathematics, not just identify the correct answer, but look at all answers in a new way, not whether they were correct, but what they told me about that child's understanding of math. Many times the model or answer illuminates a mathematical concept that is a piece of the mathematics that I have just missed.
>
> Sometimes just by writing down what they say I get more clear on what the mathematical idea is. Some days I just don't and we end with a fizzle, but I don't worry about that so much any more because I know that during the rest of the day and the evening it will usually come to me what problem to present or where to start the discussion the next day.

In a summer conversation with Lauren, Kathy talked about the skill she had learned over the course of the year: "It's learning the right question to ask. It's asking the question that will synthesize the discussion and knowing the question that will pull it together and challenge the kids in a way that will move them forward."

In the IMT Group

Even though interesting things were happening in the classroom, Kathy reported at the next meeting of the IMT group that she was feeling overwhelmed by the number and complexity of the questions the third graders were raising. Moving from fractions of a whole to fractions of a set had introduced unexpected confusions. For example, Marianne rejected the claim of a classmate that one plate was 1/8 of a set of 8 plates, asserting, "You can't have fractions when you haven't cut something up." Kathy added, "They are really pushing on this." Debi said, "What I think is, it's neat that they are pushing." After some further discussion, Kathy announced, apparently only half in jest, "I want to go back to negative numbers!"

However, an event that occurred less than an hour later in the meeting suggests that she was beginning to gain new confidence in her own ability to address mathematical questions. Lauren and two other members of the IMT group were describing what they had seen on a visit they had made to Ball's classroom earlier that day. (Although we had been watching videotapes of Ball's teaching on and off all year, this was the first time the teachers had actually visited the classroom.) Lauren mentioned a question Ball had posed to one of the students: Why do you get an even number when you add two odd numbers? She added that she was still puzzling over it herself. Kathy *explained* to her why this was so—the first time she had volunteered to *explain* a mathematical idea in the IMT group. Although her *explanation* was clear and cogent, she reported later that she had felt uncomfortable about the exchange—perhaps because she was so unaccustomed to taking the role of *mathematics teacher* with another adult.

Reflecting on Fractions

Two months later, reflecting back on what she had learned and how she had changed over the course of the previous year, Kathy noted that she thought that she had become clearer about the concepts she was trying to teach and had started to think more in terms of concepts and less in terms of activities: in the past she would begin to plan a unit by looking through books for activities and then organizing them into a logical sequence. She now starts with the ideas she wants the children to explore. This change had not occurred just over the course of the previous year but rather as part of an agenda on which she had been working for several years. However, in the previous months she felt she had made a quantum leap forward. In the fractions unit, for example, she had focused on helping the students to understand what the top and bottom numbers in a fraction mean and on the connection between fractions of a whole and fractions of a set.

She continued this reflection six months later, in a conversation with Helen, this time talking about what *she* had learned about fractions as she taught this unit:

> The kids were having trouble understanding what the top number meant and what the bottom number meant. I had never wondered, and I saw it would be important to understand it. I think, like Debi said, you learn something **solid**, like it just **is**, it's hard to pick apart. Like a fraction, 1/2 or 2/4: To me it was clear that you could have 2/4 of a pie or a rectangle. I knew you could have 2/4 of a set.
>
> I remember doing the crayon box problem, with 48 crayons, and the kids struggling with that and as they struggled I began to ask, "Oh, OK, what does the 2 mean? what does the 4 mean? I did sit down and I went to math books—like the *Standards* and I have this brown book that I use—I don't know why because it never helps me— and I think I looked at Burns and Tank.
>
> The kids teach me how to teach. I don't consider that I have a strong grip on mathematics; I was surprised, but maybe not shocked, that there was more to fractions than I had seen.

"The kids teach me how to teach." Clearly they do this in part by helping her locate the central and interesting ideas in a problem, by asking questions, and by showing her which of her own mathematical ideas she has not probed deeply. When the children raise the questions, Kathy listens carefully, pushes her ideas hard, talks to other people, and comes to new understandings of her own as well as new ways to teach.

DEBI

> How to really teach for understanding?
>
> How do I know if the students are really understanding?
>
> What does it mean to know?
>
> How to get students thinking and talking about math?
>
> How to create lessons in which there is discourse and students have tools and strategies to search out solutions and talk about their solutions?
>
> How do I find time and people to talk about math this way and not the more traditional way?
>
> How do I learn to question students and push their thinking in math and all areas?
>
> How can I learn to create my own curriculum when I am not strong in my math skills?
>
> How can I learn more about math so that I know how to take advantage of teaching situations (teachable moments)???

In October of 1991, having been a paid teacher for just one month, Debi attended the first meeting of the IMT group. Afterwards she recorded these seven questions about math and math teaching in her journal. The depth and range of the questions suggests how hard she was thinking about what she needed to know in order to teach math in ways that were different from those she had experienced as a student. Her last question sounds a particularly interesting note, for it suggests a kind of confidence about the possibilities for expanding her knowledge of mathematics which is relatively rare among teachers who have been defeated by math in elementary school and still consider themselves weak in this area

But if Debi was optimistic, in the fall of 1991, that she could "learn more about math," it was not because she had experienced more success in elementary and secondary school math classes. On the contrary: The path to this moment had been long and difficult. Five years earlier she was unsure about her ability to take college courses or contribute anything to a conversation. She had a particularly low opinion of her own capacity to learn math.

Elementary school, high school, and college

In February, 1992, Debi recalled her school and college experiences of learning math this way:

> When I was a student in elementary and high school I didn't understand math and as a result I hated it. I was taught how to do the process [algorithm] but I had no clear understanding as to why I was doing it. The teacher would give out the page numbers in our math book that were to be done and if it was a new concept she would explain the one or two examples at the top of the page. Then each child would complete the problems. The students didn't talk to each other or share ideas. The only time a student interacted with the teacher was if she asked a question or wanted an answer to the problem as we were checking the pages for correct answers later. I don't remember a teacher saying "I don't think you all have a clear understanding of this or you seem to be having problems and so we'll go back and look at this again." The next day, no matter how we did on the previous pages, we'd be off to the next page in the book. I think I knew that each page in the book was going to be covered that year and by the end of the year you could always feel a push from the teacher to "finish" the book.
>
> I always struggled to keep up and never felt I had a good understanding of math. It took me a long time to catch on to the algorithm that was being taught and so I was always behind and once you get behind in math you never seem to catch up. At least that was how I felt. These experiences created a strong case of "math anxiety" and I made every effort to avoid taking any math classes that weren't absolutely required to graduate from high school. I eventually decided that some people could "do" math and some couldn't do math. This was reinforced by a society that suggested that boys were better suited to study math and science than girls and by a father that steered me toward literature and history type courses because he thought I was better suited for those type of courses.
>
> This avoidance of taking math courses followed me into college and I was always searching for majors that didn't require any math courses. Naturally this eliminated a lot of choices in my college career. I eventually dropped out of college to get married and found myself doing basic accounting work in my job. I discovered I could understand and do math that was related to accounting, such as adding, subtracting, and percentages. It was more real world math and it seemed to make sense to me. I decided that this was the "kind" of math I could do. The other "kind" of math (and I am not sure what I would have included in that category —perhaps intellectual math) I couldn't do.

Returning to College

In 1988, having concluded that she needed to be able to make more money than she could earn as a secretary or a bookkeeper, Debi decided to return to college. She needed to take some math classes as part of her program:

> This meant I would have to take a placement test with the math department to see where I would have to begin. I dreaded this. I knew that I had very little background in math and didn't want to make a fool of myself. I went to the bookstore and bought a pre-algebra book that would allow me to teach myself math. The book took me, step by step, through different math processes with great examples. I felt like it was a challenge for me and I loved doing it. I spent the entire summer doing every problem in that book and when I took the test I was able to go into a beginner's algebra instead of pre-algebra. I felt successful in math for the first time in my life.

Her success in the first math course boosted her confidence still further: the course was self-paced; each student worked independently in a book similar to the one she had used during the summer. There were weekly meetings scheduled for those who wanted help, but Debi found that she did not need to attend them. She was delighted: "I didn't even need to go to the classes. I could *do* this on my own!" She got a 4.0 in the course—a further boost to her confidence that she could learn math after all.

She felt, however, that being able to move through the material at her own pace was essential to her success as a learner of math.

> And at the beginning, especially during that first summer, it was a very slow pace. Because I remember that when I finally hit that first class that you had to take as a *class*—because when you hit algebra and trig you are back in the real world again—I dropped the first one I took. I lasted about two or three weeks. I *could* not keep up—or didn't think I could keep up—with this guy's pace.
>
> I was really upset, dropped the course, waited, took it again next time with a different instructor. And that helped, but I still had to move at their pace and that was harder for me. I did it in the end, but it was harder.

Difficult as it was to step back into a math class where she had no control over the pace, she was convinced that she needed to prove to herself that she could do it.

> I could not let it get the better of me. I just would not let it. . . . I took two terms of economics based on the same thing: I had flunked them in college the first time around and I was not going to let it get the better of me. So even though economics did nothing for me as far as credit toward something, I took them.
>
> It had hung over my head all those years and I had to beat it.
>
> And after I conquered a little bit of math, and economics, I think I realized that I could do anything I wanted to. And then watch out!

These victories over the old demons of school mathematics seemed as crucial to her in retrospect as they seemed in prospect. When she wrote and talked about her own learning, her prose rings with the accents of celebration. But when Helen asked her, in February of 1993, whether she had always felt this way about learning she shook her head:

> I love to learn. I really do. But I still struggle with feeling dumb.
>
> But this celebration of learning came when I realized I could do it, which was when I did the math. And at the same time I was doing the math, I took a psychology class. And I got a 4.0!
>
> I can remember, in the orientation, on the first day at [the community college], they asked, "Why are you here?" And I remember saying something like, "I just need to see if I can do this, and I'm just so excited to be here." And I got a 4.0 and I wasn't dumb! I shook in my boots the whole time, but—
>
> The other piece was, when I was in my teacher ed. program, I was in a cohort [taking all teacher education classes with the same 29 people] and I finally became really comfortable with sharing my ideas. That's when I began to feel, "Well, I'm not so dumb." It was like my opinion was worth something.
>
> It always takes me a long time to warm up and say something. I am always extremely quiet at first in a new class.
>
> I'm still in the mode of thinking I'm dumb. I'll be glad if I can ever get past that.

In the Elementary Classroom

By the time she began to spend substantial pieces of time in an elementary classroom in her last year of college, Debi had come a long way in defeating her image of herself as someone who couldn't do math. She knew that, with hard work and time, she could get a 4.0 in a college math class. Looking back, however, she believes that she was still entirely reliant on memorization for this academic triumph. In a journal written in the fall of 1991 she explains a bit about the way she had thought about numbers, for example, in the early months of the 1990-91 school year:

> When I began teaching subtraction to the second graders, I had a process [algorithm] firmly in my mind. However, I knew that I wanted to teach them for understanding. I wrote my unit with that goal in mind. On paper it was teaching for understanding.

> However when I began teaching the unit it became process-oriented because it was what made sense to me. When Sharon and Kathy were talking about this with me I got very frustrated. They kept talking about numbers in parts. 4 and 3 are parts of 7. But to me it was a solid number 7 that couldn't be broken apart. Because of this idea I had of numbers I was having trouble teaching subtraction the way I knew I should because it didn't make sense to me. Eventually during that unit it began to make more sense to me . . . maybe I learned with the kids. I noticed this year that numbers don't seem so solid and I'm thinking of them as parts of numbers that can be put together.

Another quote, this time from a conversation with Helen in 2-8-93:

> That first year, watching Kathy, I couldn't get past, "Well, she is doing addition." And I didn't know what addition meant, really. . . . When I looked at 23+23, I saw the 23 as solid: It wasn't 2 tens and 3 ones.

Throughout the year Debi struggled to teach math "for understanding." There were intriguing moments in which children managed to explain *their* ideas:

> I got started with what the equal sign means. I don't know if I helped or confused them. I thought it was interesting to watch them trying to think through what it meant to them. I found it interesting to try to see how they were thinking. We came to a shared definition that the equal sign means that both sides are the same. (December 12, 1990)

Over and over, however, her journal recorded her frustrations. On the one hand, there were her goals: "I want my students to understand what they are doing. I don't want them to just memorize procedures to follow." But on the other hand there was what she saw herself actually teaching: "I do think I've been concerned with the teaching of one strategy to use. If I teach one strategy it seems to become a process" (January 1991). She connected this difficulty to the way that she had been taught math and to the fact that numbers were still "solid" for her.

Fall, 1991

In the fall of 1991, Debi took her first paid teaching job: She was a "co-teacher" at the elementary school in which she had student taught. Instead of working with one group of children all the time she taught four different primary classes, spending two hours with each group each week. She also joined the IMT group.

In the Second and Third Grades. Her math journal is a kind of celebration of her own learning: Over and over again, as she watches one of the teachers she works with teach a lesson, she sees mathematical concepts embedded in the lesson that she is certain she would not have been able to see a year earlier. For example, a week after school started she wrote:

> When I planned the Stars and Circles lesson [an activity designed by Marilyn Burns], I saw so many concepts. I could see the concept of adding groups of numbers, learning to represent numbers on paper, multiplication concepts, putting numbers in groups, and that numbers can be broken apart. . . . Last year I wouldn't have seen this.

She was also able to design a math unit of her own to give her students the kind of "feel" for metric units that had been so lacking in all of her school-based encounters with mathematical topics. In a journal written for the IMT group she explains:

> I work as a co-teacher and have the opportunity to choose what area of curriculum I want to focus on and for how long. After observing the lack of time spent on measurement last year and that often it was just memorizing I decided to choose this area for my first unit.
>
> I planned my first lesson around helping the children learn about the metric system and especially the decimeter. I gave them a "measuring stick" and asked them to go around the room and find objects that were about the same size. I didn't tell them that it was a decimeter. They were to find an object, draw the object life sized and identify it.
>
> Later as we processed the lesson I identified the name of the unit. We compared it to inches. They were sent home and asked to find five objects at home that were that size. I wanted the students to have a chance to measure and begin to recognize what a decimeter size looked like with familiar objects. By having them draw the objects it reinforced the recognition.

> From here I'll help them discover that 10 dm=1 meter and, if you divide, a dm=10 centimeters.
>
> I wanted metrics to make sense to them, for them to be able to use it and identify familiar objects as a certain length so it became a part of their experiences and knowledge. (Not just memorizing). I am attempting to make this a way to better understand metrics and measurement.

Debi's subsequent journal reflections on the lessons in this unit indicate that she felt she had succeeded in achieving many of these goals. And her reflections on her own learning contain a consistent note of celebration as she talks about the math concepts she had seen embedded in lessons she taught.

The IMT Group. Debi's reflections on the first meeting of the IMT group reflected the same excitement about her own learning: As she sat down to write after the meeting, she contrasted what she had seen in the videotape that night with what she had seen when she watched a tape of Ball teaching two and a half years earlier in her first teacher education course.

> I first saw [Ball] on tape when I was taking TE101. I was impressed with how she "let her students" teach themselves and didn't seem to have much input in the lessons except to set up the problem. She never seemed to tell them they were right or wrong. . . .
>
> When I was watching the tape Thursday night it was through more experienced eyes (though still very much the novice). . . .Where the first times I saw her tapes I thought, "What a great teacher," and couldn't go any further, this time I was able to watch to see what she was doing and ask myself why she was doing it. I was able to think about what the kids were saying and then try to decide why they said it and how they were thinking. I was also able to look at the lesson and see the many directions it was going and not that it was just a subtraction problem that they were having problems with. I was also noticing how she set the original problem up in a way that would bring out different concepts (she asked them to make a mathematical sentence that equals 10), such as addition, subtraction, multiplication, and division and probably many others that could have been brought out.

Delighting in the knowledge that she is seeing and hearing far more than she had three years earlier, Debi identified these areas of change: She was now pushing herself to make sense of the students thinking; she was now also noticing all the different mathematics embedded in the lesson—seeing more than a "subtraction problem the kids were having difficulty with."

Although Debi continued throughout the fall to feel good about both her progress in teaching for understanding—the goal she had set for herself—and her own increasing ability to hear and see new mathematical ideas in the lessons she was teaching, her initial enthusiasm for the IMT group activities quickly turned to dismay. After the group's third meeting, in which we had worked, at first individually and then collectively, on designing and then evaluating representations for teaching third graders about negative numbers, she wrote that she was feeling "very frustrated with Math class. [5] . . . These discussions don't seem interesting to me . . . they seem to drag out and go no place." She went on to explain in her journal that a part of her frustration with the group was the focus on negative integers:

> I'm also having trouble with the negative numbers. To me they seem like non-numbers [6] and how do we teach them if they don't exist. . . .
>
> When I look at/ think about negative numbers I think about the number line and it makes sense because there are rules. If you have two negatives you add them. If you have a negative and a positive and the positive is higher the answer becomes positive. (I think that is the rule). But you can see if I forget the rules I'm lost because I have no idea why it's true.

In her work with Kathy and Sharon, and then in her work with children in the classroom, Debi had found new understandings of numbers. She felt that for the first time she was beginning to understand numbers and mathematical operations: "I noticed this year that numbers don't seem so solid and I'm thinking of them as parts of numbers that can be put together." She was excited by the fact that she had come to have some understandings of things she had previously learned by rote. She was determined not to slip back into the memorization mode:

> So somehow I have to gain a better understanding of what negative numbers are if I'm to really be able to take part in this class. Why was this class focused on such a difficult concept as opposed to something more "normal"?

The IMT group confronted her with an area of mathematics that did not yield very well to efforts to connect it with concrete reality. It did not provide her with the tools or the environment that would help her to make sense of this area of mathematics. Thursday night meetings became more and more painful. In December she reflected back on her experience of the group:

> I desperately wanted to understand [negative numbers'] purpose and not just a process by which to use them. I still haven't discovered that yet. But I have been struck by the struggle I was going through and the sense of frustration I was experiencing. I didn't want to go to class. I didn't want to do the project. I really just wanted to stop coming. I began to tune out. . . . It was an old feeling (the feeling of wanting to just drop trying to understand negative numbers and feeling like a failure) that I haven't experienced in a long time. I've been used to accepting the challenge since I returned to school and not getting discouraged. But this time I did become discouraged. (December 1991 journal entry)

A year later she interpreted the encounter with negative numbers this way:

> I just stepped right back into that old mode. It was a real gut reaction. When I look back, I ask, "Why did you do that, Debi?" I don't quite understand it except to say that must be a real powerful thing in me. It was a 35 year experience, and it was sitting there underneath the surface, and, for whatever reasons, it just jumped up and grabbed me and for a brief period I was back in that *dumb* mode.

When Helen asked her why she thought she had responded so differently to the challenge of making sense of negative numbers than she had when, a year earlier, her cooperating teachers had challenged her to think differently about subtraction and the decomposition of numbers, she answered:

> For some reason it was much harder content.
>
> Partly it was the setting of a different group of people. A larger group.
>
> I was still in that concrete versus abstract and "I must be a concrete learner and I'm not an abstract learner." . . . I think that was still in the back of my head. Because I knew that the math I had learned easily in my [community college] courses was stuff that I could just memorize. . . . When it came to problems that I really had to do some delving into or thinking about how to go about doing them, those were the ones I always struggled with. And I knew that piece of me, so, of course, that was the abstract piece. So I knew that I still had some limits on math and obviously negative numbers was one of them. So I had a good excuse.
>
> I think I'll always struggle with it, but I won't let it get to me. I'll just struggle harder. . . and I think of it as a challenge.

Winter 1992

Although she was strongly tempted not to return to the IMT group after winter break, Debi decided to give it another try. And with the focus shifted away from negative numbers, she found that she enjoyed the Thursday evening meetings far more. The most important developments, however, occurred in second and third grade classrooms. Unlike most teachers, she had two different school-based sites for learning about math and about teaching.

Teaching Division. The first, of course, was her own classroom. Having finished the unit on measurement, she decided to teach the second- and third-graders a unit on division which she described in a January journal entry:

> The students began by hearing the story, *The Doorbell Rang*, by Pat Hutchins. This is a story where the mother makes 12 cookies for her son and daughter. They divide it between themselves and then the doorbell rings. The kids now have to divide the cookies between 4 people and so on. The kids retold the story in play form and physically divided the cookies (blocks). Then I passed out the cookies to each child from the tray of real cookies and ended up with some left over. As they were eating their cookies I asked them to write.

The idea for this unit came out of a book. But after teaching a few lessons she altered her plan; on January 14 she wrote:

> When I first started this unit, I saw it as a *4-6 week unit*[7] that would end when I did the last activity in the book. I would do exactly what they told me (which is fine and a good place to

start) and then the unit would end. Now I'm trying to think of a way to extend this and use it more as an introduction to division. I'm not sure how to go about this but I really want to try.

The events in the classroom led to conversations outside of the classroom:

> I talked with [a graduate assistant working in the school] later about [the different ways the children had found to share the cookies] and she was telling me about two types of division. One type she called "partitive," which she described as How many groups of something will I get? The second type she called "distributive," which was How many each box will get.

The decision to go further into division also led Debi to begin to read and think more about what division actually was:

> As I was planning [the unit focused around *The Doorbell Rang*] I said to myself that this was about division. But I didn't think about what division meant. I began teaching the unit and it began to take on a life of its own. I started thinking about going beyond the planned lessons. Then I began thinking about how I would teach division for understanding. . . . I asked myself the question What was division? What concepts were embedded in division? I pulled out my math books [8] and began researching and thinking about it. I've decided that division is the opposite of multiplication (inverse)—42÷6=7 and 6 x 7=42. Other concepts were subtraction (12÷3 can be figured out by 12-3-3-3-3 or =4) You could count backwards to get the answer 15÷3 . . . 15, 12, 9, 6, 3, 0. Five numbers. Multiplication is needed. Addition. Place value understanding. Fractions. Remainders. Decimals. There are so many things/ideas that go into division.

In March, summarizing what she saw herself learning over the previous months, Debi wrote:

> It's become very clear to me that the first step in teaching a concept to children is for me to try to understand the concept first. I get out my books and try to find out what mathematical ideas are embedded in the concept, talk with other teachers, and do actual problems myself. I'm also realizing that as I begin teaching the concept I will probably learn more from the children as they try to solve problems.

During the summer, in a conversation with Steve, she recalled how her learning continued as she began teaching:

> And [the students] taught me because when they were doing it themselves, somebody was taking the original number and subtracting it, and immediately I thought, "you can't do that." And then I started thinking, "But it works!"

> I was more open and then I started watching different ways they found to solve it.

Seeing connections among topics was exciting. In addition, it made the mathematics more interesting and accessible:

> Even your negative numbers are really so connected to subtraction and trading. And I think that is fascinating. Once I connected the negative numbers to subtraction, it made a whole lot more sense.

Some months later she discussed the pedagogical implications of discovering connections between division, subtraction, addition, and multiplication:

> Last year, when I saw what the different pieces of division were and how connected all these concepts are, I began to wonder why do we have to teach one before the other necessarily because in a way they are so [connected]. I never knew all this stuff before. It was just, if I couldn't memorize the process, I couldn't pass the class. . . . That's how I got a 4.0 in my college classes. (IMT, 1-93)

Kathy teaches about numbers below zero. Debi had a particularly strong connection with Kathy's third graders because she had student taught in Kathy's room and Kathy was keeping the second graders she and Debi had taught together for a second year, following them into third grade. As a consequence, when Kathy decided to venture into the land of negative numbers, Debi was particularly interested to see what happened. She was also quite astonished by Kathy's decision to teach this material, since she knew how little Kathy had enjoyed exploring this numeric territory in the IMT group:

> I kept popping in on Kathy's group as much as I could when she started this. I had to see how this [would go]. I couldn't believe it would work. (summer conversation with SS)

When Kathy announced her decision to do this unit, Debi wondered, "What's the point?" just as she had when she learned that Deborah Ball had taught this material to third graders. But as she observed Kathy's students, she began to find some answers to this question:

> I just kept remembering Nathaniel going, "I did it, I did it!" or something like that. And it answered his questions.
>
> And then I remember Lucas from the first year [when she and Kathy were working with the same students as second graders] saying, "Can't you do that? Isn't there . . . a number?" And Kathy and I were going, "Nope. You can't do that." And we were both looking at each other and going, "Well, what are we supposed to do?" And that year we chose to ignore it.
>
> Kathy wouldn't do that anymore. .. I don't know if she'd pursue it, but she'd give them an answer, some sort of explanation. . . . Whether she would pursue it or not, who knows, but she wouldn't want to drop it like she and I did the first time it came up.

Watching the third graders that she had known for almost two years delightedly creating numbers sentences "that end below zero," she began to see reasons for introducing them to this numeric territory. In addition, she told Steve, "Watching the kids go through it helped me figure it out more."

Before long, the second graders she was teaching asked her if there weren't numbers below zero. A year earlier, she and Kathy had ignored a question that seemed to be headed in this direction; this time she addressed it head on: "I said, 'You're absolutely right. That's called a negative number,' and I pointed to my number line. And then I said, 'You'll get into that. Talk to [your homeroom teacher] about that one.'"

Fall, 1992

Debi had come away from her experience with negative numbers in the IMT group determined that she would never again let herself slide into passivity and defeat as a learner. She had seen the danger of "going right back into that old mode," and resolved to guard against it. For that reason, an event that occurred in the third IMT meeting of 1992-93 stood out as a marker for her and for Steve, Helen, and Lauren. Another teacher was explaining an idea that had cropped up in her sixth grade math class about the division of decimals; at the IMT group's suggestion she had moved to the blackboard in order to make the idea clearer. Many of the rest of us were copying down the problem she had put up in order to think more about it. Confused by what Lisa was saying, Debi leaned over to Steve, who was seated next to her, to ask a question. As he explained what he thought Lisa's students had been saying, Debi whispered with a triumphant grin, "It's like negative numbers all over again. Only this time I'm *challenged*!"

Reflections on Three Cases

Kathy, Carole, and Debi have much in common: They are all white women, all three teach primary grades in the same urban elementary school, all studied mathematics in highly traditional elementary and secondary classrooms and all emerged from them between twenty and thirty years ago with a strong aversion to mathematics, with little experience of learning math conceptually, and with a self-definition that had "not good at math" stamped on it in bold, apparently indelible, letters. All three have worked hard to learn to teach mathematics more conceptually and to provide experiences for their own students that will promote deeper understandings of mathematics and more enthusiasm for doing mathematics. In addition, they spend considerable time talking together about teaching, children, and the puzzles of mathematics teaching. All have made important changes in their understandings of mathematics and their stance towards mathematics.

Yet despite these important similarities and shared concerns, their stories, although overlapping, are quite different. They describe different paths to learning and different outcomes. We want to look here at some of the key features of these stories. After that we will look at the learning they describe and offer some conjectures about what features of their own primary grade classrooms seemed to have fostered their own learning of mathematics so much more successfully than did a college classroom.

To us, the most dramatic feature of *Carole's story* is the language of excitement, celebration, and discovery. When she talks or writes about her own learning or about her students' ideas and insights, Carole's language fairly glitters with vivid verbs and compelling images of travel, awakening, and discovery. Math, which used to be sterile, "dry," "black and white," now "has life, it has many questions and lots of answers." The children have knowledge and ideas she never before imagined. Ideas and numbers that seemed isolated now connect in previously unimagined ways. Carole seems to be telling a story about discovering connections. She sees new connections among mathematical ideas as, for example, she watches her students create number sentences equal to ten: "I didn't realize 10 was out there all those different ways." She finds that people can connect with one another as they do mathematics and try to communicate their ideas to one another in her own classroom and in those of her colleagues. She discovers new connections between people and mathematics—it has become, for example, "very obvious" in her own life.

Her excitement about her students' ideas and the connections they are forging seem to propel her forward. It makes her want to listen to her students, to hear more of their ideas and to work to understand and appreciate them. Her discovery that strategies that she saw them using to solve a problem and dismissed—"That can't work"—were fruitful—"No, wait a minute, that *does* work!"—leads her to listen to them with faith and careful attention.

By creating an environment in which children explore and articulate mathematical ideas, and by listening to the ideas that the children then articulated, she has learned important things about the nature of mathematics and about what it means to do mathematics.

Debi's story is somewhat different. Although she was originally defeated by math in school and college, several later experiences with formal mathematics courses built up her eroded confidence. By the time she started student teaching in 1990, she had earned 4.0s in several math courses. Although she believed that her knowledge of this math was highly procedural and depended on her memory of formulas in the book, she felt that she could now handle what she called "concrete math."

The skill she had developed in her community college courses did not, however, equip her to teach math in the way she wanted to teach it. For this she had to explore numbers and basic arithmetic operations more deeply. She had to find ways to see numbers as decomposable rather than "solid." She had to learn more about the connections among arithmetic operations. Some of this she accomplished in the classroom, listening to children present ideas and stretching to understand what they were saying. But a good part of her learning came outside the classroom as she prepared to teach, as she thought through, for example, a unit on division and tried to connect it to work her students had done earlier on multiplication. The work she did outside helped her to understand their understandings—the things they said, the representations they created on the chalkboard and in their notebooks. It also led her to connect addition, subtraction, multiplication, and division in new ways—and to raise questions about the practice of teaching them in isolation from one another.

If Carole's learning was fueled by her students' newly visible ideas, Debi seems to be propelled in part by her sense of herself as a learner, her celebration at continuing to learn. If Carole's story evokes images of Dorothy opening the door on the technicolor world of Oz, Debi's suggests someone who tasted both defeat and success at the learning game and takes special delight in her own learning because of the journey that has brought her to it.

Listening is an even more central theme in *Kathy's story* than in those of her colleagues. Her first journal entry examines why students in her classroom listen best to one another during sharing time; in June she identified changes in the way she was listening to students as central to the changes she had made in her practice. It is listening to her students, in part, that carries her into the land of negative numbers. And it is listening to them that convinces her that negative numbers are not, in fact, as difficult and abstract as she had thought.

Over and over, as she watched the third graders working in this numeric territory, she shook her head disbelievingly and whispered to Helen, "I can't believe how easy this is for them."

Kathy's interest in listening connects closely to her delight in her own learning and in her students' learning. Like Carole, she is excited by their ideas, by the way that they make sense of mathematics—as well as other things. And she is delighted by the ways in which ideas in the classroom generate other ideas, by the ways in which a good question generates intellectual discourse.

Her commitment to the adventure of her own learning carried her to the IMT group; three and a half months later it led her to propose that, with help, she would like to teach her third graders about negative numbers. To someone as uncertain as Kathy was about her own grip on this content, this was a truly frightening undertaking—she recalls feeling quite panic stricken on the day when Helen took a wrong turn on the way to school, leaving her to launch the unit on her own, totally without support. And yet, as she said to Helen five months later, "I was like Henrietta Hen: I couldn't wait to get to school in the morning to teach about negative numbers."

Inside the Primary Classroom

Before they chose the unit on which the IMT group would focus during the fall, Lauren, Steve, and Helen realized that some teachers would feel intimidated by the focus on mathematical operations involving negative numbers. They believed, however, that as members of the IMT group watched eight-year-olds work with the representations that Ball used in her classroom and listened to videotaped discussions, these fears would fade (see Featherstone, Pfeiffer, & Smith, 1994). They thought, in short, that watching these videotapes and exploring the thinking of children would prove an effective way to learn mathematics. In fact, however, they were wrong: The teachers who said that they felt uneasy and unsure in this mathematical territory in October still claimed to be uncomfortable there in December. Although the explorations of the M.A.T.H. materials seems to have laid the groundwork for other important developments within the group (see Featherstone, Pfeiffer, & Smith, et al., 1993) and led to changes in the way some of the teachers taught math, it did not appear to have altered the way the teachers thought about themselves as learners of mathematics or about the specific subject matter—operations with integers.

Nonetheless, over the course of the 1991-1992 school year, Carole, Kathy, and Debi did make major changes in the ways in which they thought and felt about mathematics and in their knowledge of the subject matter. They traced most of these changes to things that happened in and around their teaching: They learned math by teaching it.

Our explorations of these three cases suggest a number of reasons why their own primary classrooms turned out to be particularly good settings for learning math—why, indeed, they were better settings than the vast majority of university classes would have been. We consider these reasons below.

The relationship between the learner and the mathematical ideas. When teachers begin to create opportunities for their students to invent new ways to solve math problems and to share their ideas and evolving theories about mathematics publicly, they are often very much excited by what they see and hear. At least, this is the experience of teachers in the IMT group, and it is an experience reported by other teachers and teacher educators as well (Schifter & Fosnot, 1993). A teacher has a special relationship with ideas generated by her own students in her own classroom. This relationship includes a sense of pride and curiosity and is different and more intense than her relationship with the ideas generated elsewhere. Thus, although Carole remembered that she sometimes lost the thread of the mathematics discussions that we observed on videotapes of Ball's third grade, she focused carefully on listening to her own students, trying to hear exactly what they were saying. All three teachers' journals are filled with excited reports of particular insights—recall, for example, Carole's excitement when one of her students explained 100-100+12=12 by saying "It's like 1-1=0 and 0+12=12." Hearing and celebrating these ideas is one of the rewards for all the work and uncertainty that efforts to teach in new ways entails. Again, Carole is eloquent on this point:

> I'm amazed at how much knowledge kids really have. I'm always amazed. I don't think a class goes by that I'm not amazed. They know *a lot*. There are a lot of them who know a lot about a lot of things. And you don't discover that unless you let them share and let them talk. (interview, January 1992)

If a teacher feels a special interest in the ideas to which she has in a sense served as a midwife—by creating the environment in which they were born—she may study them with special care, making a greater effort to understand them than she would invest in a mathematical idea she encountered in another setting. In addition, Kathy points out, as a teacher she feels a professional obligation to make every effort to understand her students mathematical ideas: "I am responsible for the children learning this. That is the bottom line and it is of utmost importance to me."

The expectations the learner brings to the setting. When Carole, Debi, and Kathy learned that they were going to encounter unfamiliar and somewhat abstract mathematics in the IMT group meetings, they did not feel very optimistic about understanding this math. Probably most American adults would have felt the same pessimism: their school encounters with mathematics have not encouraged them to believe that they will understand new mathematics even if they succeeded in getting adequate grades in mathematics courses. They bring the legacy of these school experiences with them to any formal setting in which they are students and mathematics plays a visible role.

But elementary school teachers who arrive at a university mathematics or mathematics education class expecting to be confused may feel very different in their own classrooms. In this setting, they expect to understand what is said. They do not expect their own students to formulate mathematical ideas which are beyond their own capacity to understand. And the expectation that they *can* understand what their students say may support their efforts to make sense of what students say and the representations that they create.

Once the effort has been made and the difficult new idea understood, Kathy's story demonstrates that the experience may have immense symbolic importance. " I have learned," Kathy reports after a year of listening hard to her students mathematical ideas, "that I can do really hard math by teaching it." Carole makes a similar comment: "I'm not afraid of it, because I can figure it out too." And Debi notes, "I feel challenged."

Nature of the learning opportunities. The NCTM argues that a classroom in which children are working on real problems, explaining their thinking, and generating multiple ways to look at a question creates a better environment for learning mathematics than does a conventional mathematics curriculum. Their arguments are based on recent research in cognitive psychology and on social constructivist ideas about learning. This research applies as much to the learning of adults as to elementary school children; it follows that teachers, if they are to construct understandings of mathematical ideas, need a chance to engage in mathematical discussions and to play intellectually with alternative representations. As matters now stand, they are unlikely to find these opportunities in a university math class. Having worked hard to create them in their own primary classrooms, Carole, Debi, and Kathy did find them there.

In an analysis of what teachers need to know in order to teach history in secondary schools, Wilson, Shulman, and Richert(1986) argued that in order to convey a concept to the diverse collection of students present in any secondary school classroom, teachers need to know their subject deeply enough to be able to represent it in multiple ways. One representation simply will not work for all learners. In the mathematics classes of Debi, Carole, and Kathy, both children and teachers have access to multiple representations, because classroom norms encourage students to generate and present them. These norms generate opportunities for teacher as well as students to look at an idea from multiple viewpoints.

What is taught. In the past decade, multiple voices have pointed out that what students do in traditional elementary, secondary, and college mathematics classes bears little relationship to what mathematicians do (see, for example, Ball, 1990b; Lampert, 1990; NCTM, 1989 and 1991): While mathematicians work, both alone and collectively, to solve mathematics problems for which their disciplinary community currently has no solutions, students in math classes work to memorize or understand the results of the work of mathematicians who have been dead for centuries. In the vast majority of math classes—at any grade level—students get no experience of *doing* mathematics. They do not learn that mathematics is a human construction, that learning to do math is, in large part learning to hear and make sense of what other mathematicians *think* about how to approach math problems. They learn, as Carole says she did in her years as a student, that "[m]athematics was very individualistic and dry. There was a process, you had to learn it, and you got through it."

In their own primary grade classrooms, Carole, Debi, and Kathy got to see people—seven and eight year olds—"doing mathematics." Day after day, they saw young mathematicians working together to find a way to solve a problem that made sense to all in the community. They worked on ways to communicate about mathematics that would foster shared understandings. They worked to validate conjectures, or to find counter examples. As they nurtured, presided over, and observed the work of these communities, they had the opportunity to redefine what mathematics *is*, and what it means to do mathematics. As Carole said in January 1993, "It's communication, it's discovery, it's adventure."

In their primary grade classrooms, Carole, Debi, and Kathy are having an opportunity that none of them had during the first 35 years of their lives, and that only a tiny minority of American adults will ever have: they participated in a community of mathematicians. We ought not to be surprised that this experience deepened their knowledge of the discipline.

What the learner must do in order to learn. Common sense tells us that few Americans who leave college with little knowledge of mathematics and little confidence in their ability to do complex math will deepen or extend their knowledge of the discipline. It is hard to learn *more* math as an adult, if you begin by feeling that you know very little. There are several reasons for this.

First, in the absence of compelling external incentives, most people avoid settings in which they expect they will feel uncomfortable or incompetent, and a history of bad experiences with school math will probably lead most people to expect to experience a potpourri of negative emotions in any organized math class or even in an informal setting where they are routinely expected to think about math.

Second, in most settings, one has to ask embarrassing questions in order to learn basic math. In April 1992, Kathy asked to IMT group whether one plate out of a set of 8 was 1 or 1/8. A year later she recalled, "I felt like I was taking a big risk when I asked that question. And then, everyone else *knew*; everyone else said, "Both!" like it was really obvious." Six months later she hesitated visibly before asking others in the group to explain a point that came up as we watched a videotape of a discussion in Ball's classroom. It is one thing to say, "I'm not good in math." It is something else to display your ignorance by asking a question that may turn out to be, as Kathy says, "embarrassingly elementary." In addition, when you ask someone to teach you something, the spotlight focuses on you in a way that may turn out to be very uncomfortable—the would-be explainer will keep asking whether you understand. Sometimes you say that you do just because having your understanding taken out, inspected, and evaluated at frequent intervals is intensely unpleasant.

In her own classroom, Kathy came to new understandings about fractions and negative numbers without displaying ignorance publicly or being put on the spot in the same way. She and Carole and Debi learned by listening to what children said and by thinking carefully about their claims and their representations. They found these activities deeply congenial.

The rewards for learning. Teachers in the IMT group are strongly committed to creating the best environments they can for their students to explore mathematical ideas and grow in mathematical power. Had they believed that they could have accomplished this by enrolling in a college math class, many would have struggled to overcome a natural reluctance to put themselves in a situation where they felt pessimistic about succeeding as students and enrolled. In fact, however, research does not suggest that teachers can count on learning what they need to know about mathematics to improve their teaching in such an environment (Ball & McDiarmid, 1990). Nor is this particularly surprising: Just as Debi did not find that taking algebra and trigonometry courses in college—and getting top grades in them!—helped her to think about numbers in ways that helped her to teach subtraction to seven year olds, most teachers will not find that the bowing acquaintance with limits that they get in a ten-week calculus class helps them to teach decimals or area. If a teacher's focus is on her own classroom, the rewards for learning math in a college class are, at best, far removed in time from the effort expended; at worst, they never materialize.

By contrast, the rewards for working hard to understand what your own students are saying or are likely to say tomorrow are immediate and sometimes immense. Consider, for example, the pleasure that Kathy felt when, after struggling for a moment with Noah and Jeff's assertion that "-10+-10=0," she understood both that it was incorrect and why it was incorrect and managed to formulate a question to help them to look at the problem from a new angle. (And imagine her further satisfaction when, a few minutes later, the one little girl who had previously failed to make much sense of numbers below zero came to the board and explained, clearly and cogently, why -10+-10 had to equal -20.) Outside of her own classroom it would be difficult indeed for a teacher to find such powerful and immediate rewards for her efforts to understand a mathematical idea.

Notes

[1]These materials were generated by Mathematics and Teaching Through Hypermedia (M.A.T.H.). In the summer of 1989 The National Science Foundation funded the M.A.T.H. project permitting Deborah Ball, Magdalene Lampert, and colleagues to document the teaching and learning in their classrooms. Over the course of the next academic year they videotaped many mathematics lessons, as well as interviews with students, mathematicians, and mathematics educators. They saved and reproduced all student work, including the students' math journals, homework, and tests. A team of graduate students kept fieldnotes on the mathematical and pedagogical issues raised in each lesson, and reproduced the teachers' journals, in which they recorded each day their reflections on lessons. During the following two years Ball and Lampert worked with teams of graduate students to create videodiscs that would permit prospective and practicing teachers outside of the college to explore some of these materials.

[2]Since 1989, their school, like a number of other public schools in mid-Michigan, has been linked to the Michigan State University College of Education as a part of the College's professional development school effort.

[3]The names used for students in this publication are pseudonyms.

[4]We had decided to ask the third graders to figure out what the temperature in Anchorage was at nightfall if it had been 2 degrees in the morning and had fallen by 6 degrees during the day. Instead, she asked them to figure out how much the temperature had fallen if it started at 2 degrees and ended up at -4 degrees.

[5]Although Steve, Helen and Lauren did not think of the group as a math class, it is interesting to note that, at least on this particular day, this was how Debi described it. And, indeed, the activity of creating and evaluating representations for the operation of subtracting a negative number clearly did require some understanding of mathematics.

[6]This view of negative numbers has a long and honorable history. As Barrow (1992) observes, "Negative numbers do not appear to have been generally recognized as "numbers" until the sixteenth century. Thus Diaphanous described as "absurd" equations with negative answers (p. 90).

[7]She was meeting with each group only once a week.

[8]In February, 1993, Debi elaborated on what she meant here: "I always go to the Standards, plus I have a couple of other books I use. And so I went and read what they were saying about it and then I tried to list what I was getting out of them. And I also just did a couple of problems and tried to figure out what I thought I was pulling out of it. And I also probably talked to Kathy."

References

Ball, D. L. (1990a). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal, 90* , 449-466.

Ball, D. L. (1990b). *With an Eye on the Mathematical Horizon: Dilemmas of Teaching Elementary School Mathematics* (Craft Paper 90-3). East Lansing: Michigan State University, National Center for Research on Teacher Learning.

Ball, D. L. (1992). Teaching mathematics for understanding: What do teachers need to know about subject matter? In M. Kennedy (Ed.), *Teaching academic subjects to diverse learners* (pp. 63-83). New York: Teachers College Press.

Ball, D. L. & McDiarmid, G. W. (1990). The subject matter preparation of teachers. In W. Houston, M. Haberman, and J. Sikula (Eds.) *Handbook of research on teacher education* (pp. 437-449).

Ball, D. L., & Wilson, S. W. (1990). *Knowing the subject and learning to teach it: Examining assumptions about becoming a mathematics teacher* (Research Report 90-7). East Lansing: Michigan State University, National Center for Research on Teacher Learning.

Barrow, J. D. (1992). *Pi in the Sky: Counting, Thinking, and Being* . Oxford: Clarendon Press.

Beasley, K., & Featherstone, H. (forthcoming). *"The big old conversation:" Reflections on mathematical tasks and discourse.*

Burns, M. & Tank, B. (1988). *A collection of math lessons.* Menlo Park, CA: Addison Wesley.

Cobb, P. (1989). Experiential, cognitive, and anthropological perspectives in mathematics education. *For the learning of mathematics* , *9*(2), 32-42.

Duckworth, E. (1987). Some depths and perplexities of elementary arithmetic. *Journal of mathematical behavior, 6,* 43-94.

Featherstone, H., Pfeiffer, L., & Smith, S. P. (1994). *Learning in Good Company: A report on a pilot study* (Research Report 93-2). East Lansing: Michigan State University, National Center for Research on Teacher Learning.

Featherstone, H., Pfeiffer, L., Smith, S. P., Beasley, K., Corbin, D., Derkesen, J., Pasek, L., Shank, C., & Shears, M. (1993). *"Could you say more about that?": A conversation about the development of a group's investigation of mathematics teaching*. Paper prepared for the annual meeting of the American Educational Research Association, Atlanta.

Harre, R.(1989). Metaphysics and methodology: Some prescriptions for social psychological research. *European journal of social psychology, 19* (5),439-453.

Hutchins, P. (1986). *The doorbell rang.* New York: Greenwillow Books.

Lampert, M. (1990). *Practices and problems of teaching authentic mathematics in school.* Paper presented at the International Symposium on Effective and Responsible Teaching. University of Fribourg, Switzerland.

Mathematical Sciences Education Board. (1989). *Everybody counts*. Washington, DC: Author.

Mathematical Sciences Education Board. (1990). *Reshaping school mathematics: A philosophy and framework for curriculum*. Washington, DC: Author.

McDiarmid, G. W. (1992). *The arts and sciences as preparation for teaching.* (Issue Paper 92-3). East Lansing: Michigan State University, National Center for Research on Teacher Learning.

McKnight, C. C., Crosswhite, F. J., Dossey, J. A., Kifer, E., Swafford, J. O., Travers, K. J., & Cooney, T. J. (1987). *The underachieving curriculum: Assessing US school mathematics from an international perspective* . Champaign, IL: Stipes.

National Center for Research in Teacher Education. (1991). *Teacher education and learning to teach study: Final report.* East Lansing: Michigan State University, National Center for Research on Teacher Learning.

National Commission on Excellence in Education. (1983). *A nation at risk: The imperative for educational reform* . Washington, DC: US Government Printing Office.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional Standards for Teaching Mathematics.* Reston, VA: Author.

Paulos, J. A. (1988). *Innumeracy: Mathematical illiteracy and its consequences.* New York: Hill and Wang.

Shifter, D. & Fosnot, C. T. (1993). *Reconstructing mathematics education: Stories of teachers meeting the challenge of reform.* New York: Teachers College Press.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15* (5), 4-14.

Stevenson, H. W. (1992). *The learning gap: Why our schools are failing and what we can learn from Japanese and Chinese education.* New York: Summit.

Stigler, J. & Baranes, R. (1988). Culture and Mathematics Learning. *Review of Educational Research,* 15.

Wertsch, J. (Ed.)(1985). *Culture, communication, and cognition: Vygotskian Perspectives.* New York: Cambridge University Press.

Wilcox, S. K., Schram, P., Lappan, G., & Lanier, P. (1991). The role of a learning community in changing preservice teachers' knowledge and beliefs about mathematics education (Research Report 91-1). East Lansing: Michigan State University, National Center for Research on Teacher Learning.

Wilson, S. M., Shulman, L. S., & Richert, A. E. (1987). "150 different ways" of knowing: Representations of knowledge in teaching. In J. Calderhead (Ed.), *Exploring teachers' thinking* (pp. 106-124). London: Cassell.

