WITH AN EYE ON THE MATHEMATICAL HORIZON: DILEMMAS OF TEACHING ELEMENTARY SCHOOL MATHEMATICS[1]

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> We begin with the hypothesis that any subject can be taught effectively in some intellectually honest form to any child at any stage of development. It is a bold hypothesis and an essential one in thinking about the nature of a curriculum. No evidence exists to contradict it; considerable evidence is being amassed that supports it. (Bruner, 1960, p. 33)

The tendrils of this famous passage still wind around current discourse about the improvement of instruction. This paper revisits Bruner's oft-quoted assertion that "any subject can be taught effectively in some intellectually honest form." While my aim is not to suggest that he was wrong, I seek to persuade the reader that figuring out what it might mean to create a practice of teaching that is "intellectually honest" is a project laden with thorny dilemmas and that teachers need to be prepared to face off with the uncertainties inherent in the goal. The new mathematics, science, and history curricula that swept the United States during the 1960s in the wake of Bruner's hypothesis gave us ample evidence that acting on his claim is not easy. This paper takes up the challenge in the particular context of elementary school mathematics: How can and should mathematics as a school subject be connected with mathematics as a discipline?

Much current educational discourse centers on the importance of teachers' subject matter knowledge (see, for example, Al-Sneineh, 1988; Ball, 1988, 1990, in press; Even, 1989; Grossman, 1988; Hashweh, 1987; Shulman, 1986, 1987; Wilson, 1988, 1990). Complementing concerns for subject matter knowledge is energetic interest in developing and studying alternative pedagogies——practices founded on metaphors such as "cognitive apprenticeship" (Collins, Brown, and Newman, 1989) and "situated cognition" (Brown, Collins, and Duguid, 1989). Both strands of the discourse are threaded with rhetoric about "understanding," "authenticity," and "genuineness,"——about building bridges between the experiences of the child and the knowledge of the expert. Teaching and learning would be improved, so the argument goes, if classrooms were organized to engage students in authentic tasks, guided by teachers with deep disciplinary understandings. Students would conjecture, experiment, and make arguments; they would frame and solve problems; they would read, write, and create things that mattered to them. Teachers would guide and extend students' intellectual and practical forays, helping them to extend their ways of thinking and what they know as they develop disciplined ways of thinking and encounter others' texts and ideas. Such themes in the current discourse echo those of Bruner and his contemporaries (e.g., Schwab, 1961/1974) as well as of John Dewey (e.g., 1902, 1916) before them. Dewey's formulations are as elegant as any of the contemporary versions:

> Abandon the notion of subject-matter as something fixed and ready-made in itself,

> outside the child's experience; cease thinking of the child's experience as also something hard and fast; see it as something fluent, embryonic, vital; and we realize that the child and the curriculum are simply two limits that define a single process. Just as two points define a straight line, so the present standpoint of the child and the facts and truths of studies define instruction. It is continuous reconstruction, moving from the child's present experience out into that represented by the organized bodies of truth that we call studies. (Dewey, 1902, p. 11)

In the last few years, several documents have appeared that sketch needed reforms in the curriculum and pedagogy of school mathematics (California State Department of Education, 1985; National Council of Teachers of Mathematics [NCTM], 1989a, 1989b; National Research Council, 1989, 1990). Their shared vision rests solidly on mathematics as a discipline as a foundation for worthwhile teaching and learning of school mathematics. Rather than acquiring basic operations and terminology, "doing mathematics" is heralded as central (NCTM, 1989a, p. 7). Students should learn to look for patterns and frame problems (National Research Council, 1990), to "explore, conjecture, and reason logically" (NCTM, 1989a, p. 5), and to engage in mathematical argument within a community in which standards of mathematical evidence form the basis for judging correctness (NCTM, 1989b).

My questions about connecting mathematics as a school subject with mathematics as a discipline are nested within this broader intellectual context. Like many others, I have assumed that teachers who understand subject matter deeply are better equipped to help students learn with understanding a mathematics that has both personal and disciplinary integrity and worth. I have worried about the problem of helping teachers transcend their own experiences with mathematics and with the teaching and learning of mathematics in order to create new practices of mathematical pedagogy. Such practices, I assumed, would draw on serious attention to mathematics and mathematical practice. Instead of doing worksheets and memorizing facts, children would engage in serious mathematics——problems and discussions, investigations and projects. Their activities would be more like what mathematicians do (Lampert, 1990b)——making conjectures, investigating patterns, modeling and representing real-world problems, and making mathematical arguments within a community (Putnam, Lampert, and Peterson, 1990). Schwab's (1961/1974) argument——that pedagogy must be based on the structures of the discipline in order to avoid corrupting or distorting its content——compellingly complements Bruner's (1960) vision of "intellectual honesty."

I have been investigating firsthand some possible ways in which mathematics as a discipline might appropriately shape the curriculum and pedagogy of school mathematics. Using myself as the object and tool of my inquiry, I teach mathematics daily to a heterogeneous group of third graders at a local public elementary school. Many students are from other countries and speak limited English; the American students are diverse ethnically, racially, and socioeconomically, and come from many parts of the United States. Sylvia Rundquist, the teacher in whose classroom I work, teaches all the other subjects besides mathematics. She and I meet regularly to discuss individual students, the group, what each of us is trying to do, and the connections and contrasts between our practices. We also spend a considerable amount of time discussing and unpacking mathematical ideas, analyzing representations generated by the students or introduced by me, assessing the roles played by me and by the students in the class discussions, and examining the children's learning.

Every class is audiotaped and many are videotaped as well. I write daily in a journal about my thinking and work, and students' notebooks and homework are photocopied. Students are interviewed regularly, sometimes informally, sometimes more formally; sometimes in small groups and sometimes alone. We have also experimented with the methodology of whole-group interviews. I give quizzes and homework that complement interviews and classroom observations with other evidence of students' understandings. This paper draws on these data from my teaching during 1989-1990.

Among my aims is that of developing a practice that respects both the integrity of mathematics as a discipline *and* of children as mathematical thinkers. Three components of mathematical practice frame my work: the content, the discourse, and the community in which content and discourse are intertwined. Students must learn mathematical language and ideas that are currently accepted. They must develop a sense for mathematical questions and activity. They must also learn how to reason mathematically, a process which includes an understanding of the role of stipulation and definition, representation, and the difference between illustration and proof (Kitcher, 1984; Putnam, Lampert, and Peterson, 1990). Schoenfeld (1989) argues:

> Learning to think mathematically means (a) developing a mathematical point of view——valuing the process of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade, and using those tools in the service of understanding structure——mathematical sense-making. (p. 9)

Because mathematical knowledge is socially constructed and validated, sense making is both individual and consensual. Drawing mathematically reasonable conclusions involves the capacity to make mathematically sound arguments to convince oneself and others of the plausibility of a conjecture or solution (Lampert, 1990a). It also entails the capacity to appraise and react to others' reasoning and to be willing to change one's mind for good reasons. Thus, community is a crucial part of making connections between mathematical and pedagogical practice.

I take a stance of inquiry toward my practice, working on the basis of conjectures about students and understandings of the mathematics; in so doing, both my practice and my understandings develop. In the development and study of practice, I seek to draw defensibly on what is good and desirable in mathematics in the service of helping nine-year-old children learn. In so doing, I necessarily make choices about where and how to build those links and on which aspects of mathematics to rest my practice as a teacher. With my ears to the ground, listening to my students, my eyes are focused on the mathematical horizon. This paper explores the tensions I experience as I face this challenge.

A Restatement of the Challenge

Bruner (1960) argues that children should encounter "rudimentary versions" of the subject matter that can be refined as they move through school. This position, he acknowledges, is predicated on the assumption that "there is a continuity between what a scholar does on the forefront of his discipline and what a child does in approaching it for the first time" (pp. 27-28). Similarly, Schwab (1964/1971) outlines a vision of the school curriculum "in which there is, from the start, a representation of the discipline" (p. 269), in which students have progressively more intensive encounters with the enquiry and ideas of the discipline. But what constitutes a defensible and effective "rudimentary version"? And what distinguishes intellectually honest "fragments of the narrative of enquiry" (Schwab, 1961/1974) from distortions of the subject matter?

Wineburg (1989) points out that "school subjects have strayed too far from their disciplinary referents" (p. 8). I agree. Still, trying to tie school subjects to the disciplines is neither straightforward nor without serious conceptual and philosophical problems (Palincsar, 1989). Before considering the dilemmas that emerge in my own efforts to teach third grade mathematics, I examine three problems inherent in attempting to model classrooms on ideas about "authentic" mathematical practice, problems that persuade me to avoid the term "authentic" in this context.

First, constructing a classroom pedagogy on the discipline of mathematics would be in some ways inappropriate, even irresponsible. Mathematicians focus on a small range of problems, working out their ideas largely alone. Teachers, in contrast, are charged with helping *all* students learn mathematics, in the same room at the same time. The required curriculum must be covered and skills developed. With 180 days to spend and a lot of content to visit, teachers cannot afford to allow students to spend months developing one idea or learning to solve a certain class of problems. And the best and most seemingly talented must not be the only students to develop mathematical understanding and insight.

Moreover, certain aspects of the discipline would be unattractive to replicate in mathematics classrooms. For instance, the competitiveness among research mathematicians——competitiveness for individual recognition, for resources, and for prestige——is hardly a desirable model for an elementary classroom. Neither is the aggressive, often disrespectful, style of argument on which much intradisciplinary controversy rests (Boring, 1929).

Finally, in any case, modeling classroom practice on "the" discipline of mathematics is, of course, impossible. As Schwab (1964/1971) points out, disciplines have multiple structures; these structures are also not easily uncovered. No one "knows" the structures of mathematics; there is no single view of "what mathematics is." My work, therefore, aims to create and explore practice that tries to be intellectually honest with both mathematics and the child. In this paper I examine Bruner's hypothesis about "intellectual honesty" by presenting and analyzing three dilemmas I encounter in trying to create a practice of mathematics teaching that is defensibly grounded in mathematics.

The three dilemmas arise out of the contradictions inherent in weaving together respect for mathematics with respect for students in the context of the multiple purposes of schooling and the teacher's role. Teachers are responsible for helping each student learn particular ideas and procedures, accepted tools of mathematical thought and practice. However, a view of mathematics that centers on learning to think mathematically suggests that the teacher should not necessarily show and tell students how to "do it"; such a view suggests that they should instead learn to grapple with and solve difficult ideas and problems. Yet creating such learning experiences may create frustration and surrender rather than confidence and competence. Fostering a classroom mathematical community in the image of disciplinary practice may lead students to become confused——or to invent their own, nonstandard mathematics. The teacher thus faces contradictory goals. As Lampert (1985) writes, "the juxtaposition of responsibilities that make up the teacher's job leads to conceptual paradoxes" (p. 181) with which the teacher must grapple, and for which there are not "right" choices. This is because the teacher "brings many contradictory aims to each instance of her work, and the resolution of their dissonance cannot be neat or simple" (p. 181). In trying to teach mathematics in ways that are "intellectually honest"——to the content and to students——I find myself frequently facing thorny dilemmas of practice. In this paper, I present and explore three such dilemmas. Rooted in the three components of mathematical practice which frame my work, one dilemma centers on representing the content, another on respecting children as mathematical thinkers, and the third on creating and using community.

Dilemma #1: Representing the Content: The Case of Teaching Negative Numbers

What concerns [the teacher] is the ways in which the subject may become part of

> experience; what there is in the child's present that is usable in reference to it; [to] determine the medium in which the child should be placed in order that growth might be properly directed. (Dewey, 1902, p. 23)

How can nine-year-olds be engaged in exploring measurement, addition and subtraction, fractions, probability? What are the hooks that connect the child's world with particular mathematical ideas and ways of thinking? Shulman and his colleagues (e.g., Shulman, 1986, 1987; Wilson, Shulman, and Richert, 1987) have charted brave new territory with the concept of *pedagogical content knowledge*, "the most powerful analogies, illustrations, examples, explanations, and demonstrations——in a word, the ways of formulating and representing the subject that make it comprehensible to others" (Shulman, 1986, p. 9). Shulman argues that a teacher must have "a veritable armamentarium of alternative forms of representation" (p. 9); moreover, the teacher must be able to "transform" his or her personal understandings of the content. In Dewey's (1902) terms, "to see it is to psychologize it." Figuring out powerful and effective ways to represent particular ideas implies, in balanced measure, serious attention to both the mathematics and the children. This is more easily said than done. I will illustrate this with an account of my struggles to find a way of helping my third graders extend their domain from the natural numbers to the integers.

Rationale: Why teach integers? When there is so much to cover, why is this worthwhile? Why is teaching third graders about negative numbers even an appropriate aim? The justifications, I would argue, are both experiential and mathematical. Children who live in Michigan know that there are a few days every winter when the temperature is "below zero"——and that means that it is too cold to go outside for recess. Many have also had experience with owing someone something or being "in the hole" in scoring a game; conceptually, these are experiences with negative numbers that have not been symbolically quantified. Still, the students assert with characteristic eight-year-old certainty that "you can't take 9 away from 0." That third graders——indeed, many older children——think the "lowest number" is zero seems problematic. Teaching them about negative numbers is an attempt to bridge their everyday quantitative understandings with formal mathematical ones.

Analyzing the content and thinking about learners. I began my pedagogical deliberations by reviewing the various models for negative numbers that I knew. The school district's curriculum (*Comprehensive School Mathematics Program*[3]) uses a story about an elephant named Eli who has both regular and magic peanuts. Whenever a magic peanut and a regular peanut are in his pocket at the same time, they both disappear (i.e., $-1 + 1 = 0$). This representation did not appeal to me, although I was sure that it would be fun and engage the children. I was concerned about the messages entailed in fostering "magical" notions about mathematics——peanuts just disappearing, for example——

[3]*Comprehensive School Mathematics Program* (*CSMP*) is an innovative mathematics curriculum which was developed by CEMREL (1981) for elementary classrooms. See Remillard (1990) for a comparison of *CSMP* with other elementary mathematics programs.

because of the widespread tendency to view mathematics as mysterious and beyond sense or reason. After considering other models such as money (and debt), a frog on a number line, and game scoring, I decided to use a building with many floors both above and below ground (see Figure 1). As is the case in many other countries, the ground floor is called the 0^{th} floor. This was an adaptation of a model called the "Empire State Building" that appears briefly for one lesson in *CSMP*.

Why did I settle on this admittedly fantastic model? Analyzing negative numbers and operations with them, I saw that there were at least two important dimensions:

- Negative numbers can be used to represent an *amount of the opposite of something* (e.g., -5 can represent a $5 debt, the opposite of money);

- Negative numbers can be used to represent a *location relative to zero* (e.g., -5 can represent a position that is 5 units away from zero).

A negative number thus has two components: *magnitude* and *direction*. Attending to the magnitude component leads to a focus on *absolute value*. This component emerges prominently in many everyday uses of negative numbers (e.g., debt, temperature). Thus, comparing magnitudes becomes complicated. There is a sense in which -5 is *more than* -1 and *equal to* 5, even though, conventionally, the "right" answer is that -5 is *less than* both -1 and 5. This interpretation arises from perceiving -5 and 5 as both five units away from zero, and -5 as more units away from zero than -1. Simultaneously understanding that -5 is, in one sense, more than -1 and, in another sense, less than -1 is at the heart of understanding negative numbers.

Just before beginning to work with negative numbers, I wrote in my journal:

> I'm going to try the elevator model because its advantages seem to outweigh its disadvantages. It's like the number line in that it, too, is a positional model. The "up" and "down" seem to make sense with addition and subtraction: Artificial rules don't have to be made. And I think I might be able to use it to model adding and subtracting negative numbers as well as positives: When a person wants to add more underground floors, that would be *adding* negatives. If someone wants to demolish some of the underground floors, that would be *subtracting negatives*. But could something like 4 - (-3) be represented with this model? I'm not sure. I guess it would be like *moving away from the underground floors*, but . . . I don't like the money or game models right now because they both seem to fail to challenge kids' tendency to believe that negatives are the same as zero (owing someone five dollars——i.e., -5——seems the same as having no money).

In this journal entry, I was settling on the building representation after weighing concerns for the essence of the content, coupled with what I knew to expect of eight-year-olds' thinking——for instance, that they tend to conceive negative numbers as just equivalent to zero. I hoped that this clearly positional model would help to deflect that tendency. I was aware from the start that the model had mathematical limits——for example, its capacity to model the subtraction of negative numbers.

We began work with the building by labeling its floors. I was pleased to see that the students readily labeled the underground floors correctly. I used the language implied by the building: We had floors below the ground, sometimes referred to as "below zero."[4] We had floors above the ground, sometimes referred to as "regular floors." The unconventional system——with 0 as the ground floor——did not seem to confuse the students, who were as a group relatively unfamiliar with multistory buildings of any kind. I introduced little paper people who rode the elevator in the building:

> ***Take your person and put her on any floor. Have her take the elevator to another floor and then write a number sentence to record the trip she took.***

Thus, if a person started on the fourth floor came down 6 floors, we would record: 4 - 6 = -2. If a person got on at the second floor below ground and rode up 5 floors, this would be written as -2 + 5 = 3. I introduced these conventions of recording because I wanted to convey that mathematical symbols are a powerful way of communicating ideas, a consistent theme in my goals.

We worked on increasingly complicated problems with the building, for example:

[4]The ^ marks above the numerals on the building picture replaced the traditional negative sign (-). This is a convention from *CSMP*, the curriculum used in many of the other classrooms in this school district. The rationale for substituting the ^ for the minus sign is to focus children on the idea of a negative number as a *number*, not as an *operation* (i.e., subtraction) *on* a positive number.

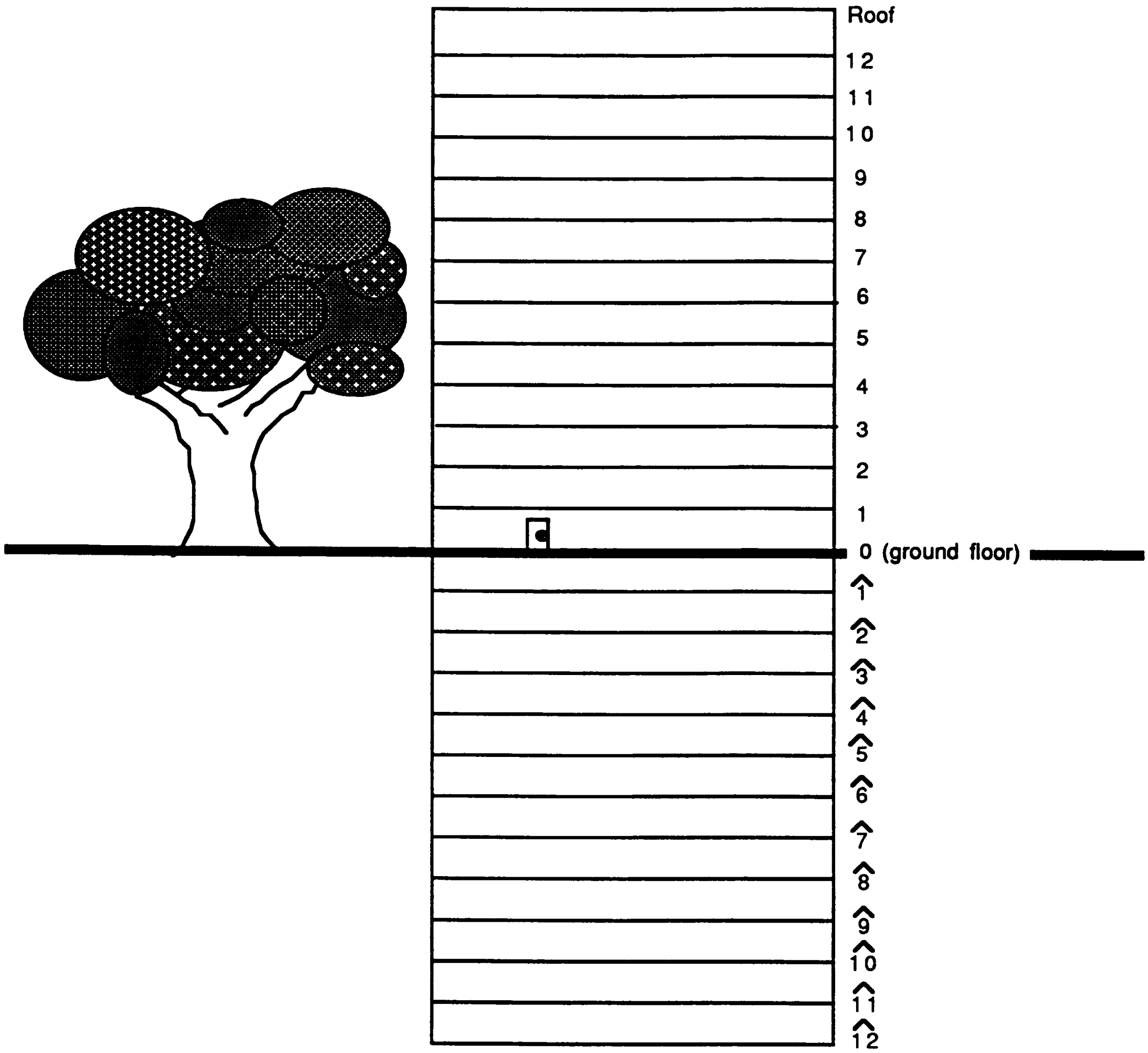


Figure 1. The "building": A model for adding and subtracting integers.

How many ways are there for a person to get to the second floor?

This problem generated an intense discussion. Some children negotiated long, "many-stop" trips for their little paper people, such as -5 + 10 - 6 - 4 + 3 + 6 - 2 = 2. Others stuck with "one-stop" trips, such as -3 + 5 = 2. The students debated: Were there infinite solutions? Or exactly 25 solutions? This argument afforded me the opportunity to talk about the role of *assumptions* in framing and solving problems. Those who assumed one-stop trips were right when they argued that there were exactly 25 solutions to this problem. Our arguments about this evolved quickly from one child's proposition that there were 24 solutions——she argued that there were 12 floors above and 12 floors below zero——to another child's observation that the ground floor offered one more solution: 0 + 2 = 2. However, those who assumed that trips could be as long as you like, were also right when they argued that the problem might have "afinidy" or maybe 8,000,000,000,000,000,000,000 solutions. As usual, the third graders reached out to touch the notion of the infinite with great fascination——and "afinidy" and 8,000,000,000,000,000,000,000 are virtually equivalent when you are eight or nine.

The work with the building generated other wonderful explorations. Nathan[5] noticed that "any number below zero plus that same number above zero equals zero" and the children worked to prove that his conjecture would be true for all numbers. Ofala produced "any number take away double that number would equal that same number only below zero" (example: 5 - 10 = -5).

Despite much good mathematical activity, such as our discussions of the number of solutions and the conjectures of Nathan and Ofala, I worried about what the children were learning about negative numbers and about operations with them. Writing the number sentences seemed somewhat perfunctory. I was not convinced that recording the paper people's trips on the elevator was necessarily connecting with the children's understandings of what it means to add or subtract with integers. I also saw that only partial meanings for addition and subtraction were possible with this model. For addition, we were only able to work with a change model of addition (i.e., you start on the third floor and you go up two floors——your position has *changed* by two floors). For subtraction, we could model its comparison sense, but not the sense in which subtraction is about "taking away." I also thought that the building was not helping students develop a sense that -5 was *less than* -2. Although being on the fifth floor below ground was

[5]All names used are pseudonyms and are drawn appropriately, to the extent possible, from the individual children's actual linguistic and ethnic backgrounds.

lower than the second floor below ground, it was not necessarily *less*. I wondered, not for the first or last time, about the relative value of using and "milking" one representation thoroughly versus introducing multiple representations.

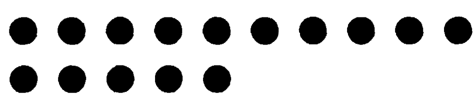
Finally, we hit a crisis——over what to do with 6 + (-6). There was no sensible way to deal with this on the building. If a person began at the sixth floor above the ground, what would it mean to go up "6 below-zero floors"? The children struggled with trying to make sense:

> Betsy: Here's how I do it. (She put a person on the sixth floor and on the sixth floor below ground and moved them toward each other.) 1 . . . 2 . . . 3 . . . 4 . . . 5 . . . 6 And so I move them both at the same time. And I got 0.
>
> Sean: But it says *plus*, not minus!
>
> Betsy: But you're minusing!
>
> Riba: (to Betsy) Where'd you get the minus?
>
> Sean: You should just leave it alone. You can't *add* 6 below zero, so you just leave it. Just say "good-bye" and leave it alone and it is still just 6.
>
> Mei: But this 6 below zero would just disappear into thin air!
>
> Sean: I know. It would just disappear because it wouldn't be able to *do* anything. It just stays the same, it stays on the same number. Nothing is happening.

Trying money. After some deliberation, I decided to try money as a second representational context for exploring negative numbers. Money had some advantages that the building lacked: Money was not positional and seemed as though it would work better for modeling relative quantities——that -5 was less than -2, despite the fact that 5 was *more than* 2. Moreover, all meanings for addition and subtraction were possible. 6 + -6 could have meaning: having six dollars and also owing six dollars. Still, I saw potential problems on the horizon. As I wrote in my journal,

> One keeps bumping into the absolute value aspect of negative numbers——for example, $-5 (five dollars of debt) is *more* debt than $-2. You have to talk about how much *money* (or "net worth") in order to make it focus on negative numbers being *less*.

I struggled with the language "props" that would structure the fruitful use of money as a representation for negative numbers. I decided that I needed an eight-year-old's version of "net worth" so as to focus the children on the inverse relationship between debt and money, on financial *state* rather than on *actions* of spending or getting money. Our first money problem was very structured as I tried to create the representational context. Instead of our usual pattern of some small group or independent work followed by a whole-class discussion, we discussed this one together:

> **Miss Suzuka has 4¢.** (*represented two ways: "4¢" and with 4 magnetic checkers*)
>
> 
>
> **She wants to buy a pencil that costs 10¢.** (*She needed—?*)
>
> **She borrowed 6¢ from Mrs. Rundquist. Mrs. Rundquist gave her an I.O.U. for 6¢.** (*written as "-6¢" and also represented with six "negative" checkers*)
>
> 
>
> **Later, she was lucky and got an envelope with 15¢ in it.**
>
> 
>
> **She had to pay back Mrs. Rundquist. What did she have for herself then?**

To resolve the debt, I had the students pay off 1¢ of what was owed at a time, matching one negative checker (representing 1¢ of debt) with one regular checker (representing 1¢). We arrived at the answer of 9¢ without much difficulty. When I wrote the 6¢ debt as -6, I asked, "Why do you think I wrote 6 below zero?" It seemed that the students found it sensible.

But when I asked them to write a number sentence to represent the story, they wrote 15 - 6 = 9, not using any negative numbers at all. And that made sense when I thought about it:

Their number sentences represented the *action* of paying off the debt (i.e., you take 6¢ from your 15¢ and give it to the person you owe). I realized that I would need to structure the use of this representation to focus on *how much money there was*, rather than on *actions*. I decided to do another problem in which the key question would be how much money Miss Suzuka had *for herself* at any given point. So when she owes Mrs. Rundquist $10 and also has $13 in her pocket, one can ask "What does Miss Suzuka have for herself right now?" and that that would support the use of negative numbers——that is, -10 + 13 = ? I conceived the idea of "for herself " as a representation of net worth; I hoped it would focus the children on balancing debt with money in ways that would illuminate positive and negative numbers. I felt that, if I could get it to work, money would be a good complement to our work with the building.

I realized as we continued, though, that the children did not necessarily reconcile debt with actual money, that they were inclined to remember both but to keep them separate. For example, if I talked about Jeannie having $4 in her pocket and owing $6 to her mother, they were not at all disposed to represent her financial state (how much money she had for herself) as $-2. Instead, they would report that "Jeannie has $4 and she *also* owes her mother $6." With money, they seemed to avoid using negative numbers——maybe precisely because the representation entails quantity, not position. As Jeannie argued, quite rightly, "There is no such thing as below-zero dollars!"

On the building negative numbers seemed sensible to denote different positions relative to the ground. But on the building we only used negative numbers in the first or answer positions of the number sentences: ___ + 2 = ___, because we never figured out what it would mean to move a negative amount on the building. With money, many children never really used negative numbers to represent debt: They were inclined to report that someone had "$6" and "also owes so-and-so eight dollars," rather than using $-8 to represent the debt. They were also inclined to leave positive values (money) and negative ones (debt) unresolved.

Students' learning. Uncertain about where we were and where we could get to, I gave a quiz. I found that, after exploring this new domain via the representations of the building and money, all 19 students were able to compare integers, for example,

$$-35 < 6$$
$$6 > -6$$

and explain why (e.g., "-35 is below zero and 6 is above zero so -35 is less than 6"). They were also all newly aware that there is no "smallest number." However, about half of them, when asked for a number that was *less than* -4, produced one that was more (e.g., -2).[6] Focused only on the magnitude of the number, -2 seemed less than -4. As I thought about how wary some of them still were of "these" numbers below zero, I reminded myself that it took over a thousand years for negative numbers to be accepted in the mathematical community——due principally to their fundamental "lack of intuitive support" (Kline, 1970, p. 267). Why should I expect my third graders to be quicker to accept a difficult idea?

Dilemmas of representation. Clearly, the representation of negative numbers is fraught with dilemmas. I had to think hard about "numbers below zero." And as I did so, I realized how rare such content analyses are for any of the topics typically taught. Moreover, the children's understandings and confusions provided me with more information with which to adapt my choices, yet the mathematics helped me to listen to what they were saying. Thus, it was in the ongoing weaving of children and mathematics that I constructed and adapted my instruction.

My analysis made me aware of how powerful the absolute value aspect of integers is——that is, that -5 is in many ways *more than* 2: It is farther from zero than 2 is. Moreover, -5 is also *equal to* 5 in some senses: they are equidistant from zero. So, given this insight, I faced the dilemma of what I should try to get my students to learn: Could they learn to manage simultaneously the sense in which -5 is more than 3 *and* the sense in which it is less than 3? In school, they will be required to say that -5 is less than 3. Am I confusing them when I allow them to explore the double-edged meaning of a negative number and what it represents?

I also had to think about what eight-year-olds could stretch to understand. Although there is research on student thinking, it has not investigated many topics that teachers teach: What are eight-year-olds' conceptions of proof and of what makes something true——in different domains? Our knowledge about primary grade children's solving of arithmetic word problems or of place value, for instance, does not necessarily help us to predict how they understand the notion of numbers below zero or the relationship between positive and negative integers.

Constructing good instructional representations and figuring out how to use them well are not the same thing. Even after I had developed these two models for negative numbers and analyzed them with respect to the mathematics and to each representation's accessibility for students, I still had to figure out how to use them. I struggled with questions about what kinds of problems to work on while using each tool and what should be the supporting language that would structure and focus the representation's key features for illuminating the content. I discovered, for example, that I needed some kind of notion of "net worth" in order to steer the

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[6]Note that producing a number *less than* -4 requires a still more solid understanding of negative numbers than comparing a negative with a positive number, as in the examples above of 6 and -6 or -35 and 6.

children's use of the money model away from attention to *actions*——buying (subtraction) or earning (adding)——to attention to *balances* and states. If I wanted to create a need for the students to use negative numbers to represent quantities, then how money was engaged as a representational context was crucial.

There are no "perfect representations," nor are there formulas for generating representations. Good teachers must have the capacity or be provided with the support to unpack and analyze the content. Thoughtful consideration of students' current ideas and interests must be threaded throughout this process of unpacking and analyzing. Teachers must also figure out how to support and use the representational contexts that they construct. In addition, teachers need alternative models to compensate for the imperfections and distortions in any given representation (Ball, 1988). When Bruner (1960) argues that constructing "intellectually honest" representations "requires a combination of deep understanding and patient honesty to present physical or other phenomena in a way that is simultaneously exciting, correct, and rewardingly comprehensible" (p. 22), he is saying a mouthful. As I try to do what Bruner suggests, I struggle with dilemmas and unanswered questions. And, if all the uncertainties were not enough, I face persistent uncertainties about what sense my students are making and about what they are learning.

Dilemma #2: Respecting Students as Mathematical Thinkers: The Case of "Sean Numbers"

Good teachers "respect" children's thinking. They view students as capable of thinking about big and complicated ideas, although what that actually means in mathematics is, at times, not always clear. Mathematics is, after all, a domain in which there *are* "right answers." Respecting children as authors or artists seems somehow different. The mathematics teacher must respect students' thinking even as she helps them to acquire particular tools, concepts, and understandings and as she strives to enculturate them into the discourse of mathematics. Hawkins (1972) captures some of this tension when he writes that the teacher must be able to "sense when a child's interests and proposals——what I have called his trajectory——are taking him near to mathematically sacred ground" (p. 113). He continues:

> A teacher-diagnostician must map a child's question as much as his answer, neither alone will define the trajectory; and he must be prepared to anticipate something of what the child may encounter farther along the path. (p. 113)

Rationale: Why teach invention? When my students excitedly noticed that only some numbers could be formed into squares out of the ceramic tiles we were using to explore multiplication and division (e.g., 9 tiles could be arranged as a 3 x 3 square, 16 as a 4 x 4 square, but neither 10 nor 12 could be arranged as squares), and that many of the odd numbers yielded only two different rectangles, they were reaching out to square and prime numbers. They were also reaching out to a kind of mathematical thinking: seeing patterns and conjecturing about their generalizability. Riba suggested that there would be more odd than even numbers that could be made into squares; however, Betsy countered Riba's suggestion, pointing out the even-odd pattern in the squares they had found thus far (1, 4, 9, 16, 25, 36, 64). The class was shocked when Jeannie and Sheena announced that, "you can't *prove* that an even number plus an odd number would always be an odd number——because numbers go on forever and so you can't check every one."

Mei: (pointing at the "theorems" posted above the chalkboard) Why did you say *those* were true?

Sheena: She just *thought* of it today.

Ofala: I think that an even plus an odd will always equal an odd because I tried . . . *(counting in her notebook)* . . . 18 of them and they always came out odd.

Jeannie: But how do you know it will *always* be odd?

Third graders tread frequently on "mathematically sacred ground." They also tread on mathematically uncharted ground. Surely "respecting children's thinking" in mathematics does not mean ignoring nonstandard insights or unconventional ideas, neither must it mean correcting them. But *hearing* those ideas is challenging. For one thing, teachers must concentrate on helping children acquire standard tools and concepts——the ideas of our mathematical heritage. As a consequence, however, the unusual and novel may be out of earshot. For another, making sense of children's ideas is not so easy. Children use their own words and their own frames in ways that do not necessarily map into the teacher's ways of thinking. Both Dewey (1902) and Hawkins (1972) suggest that a teacher's capacity to *hear* children is supported by a certain kind of subject matter knowledge. Hawkins describes it:

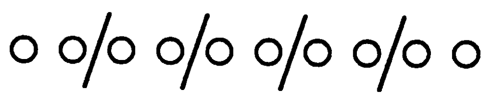
> A teacher's grasp of subject matter must extend beyond the conventional image of mathematics. . . . What is at stake is not the . . . end-product that is usually called mathematics, but . . . the whole domain in which mathematical ideas and procedures germinate, sprout, and take root, *and* in the end produce visible upper branching, leafing, and flowering. (Hawkins, 1972, p. 114)

So, even when the teacher *hears* the child, what is she supposed to do? What does it mean to respect children's thinking, while working in a specialized domain that has accepted ways of reasoning and working and accepted knowledge (Kitcher, 1984)? In the situation described below I explore this problem in the context of one child's unconventional idea, an idea I chose to extend and develop in class.

Appreciating the mathematics in the child. After we had been working with patterns of odd and even numbers, as we began class one day, Sean announced that he had been thinking that 6 would be an odd *and* an even number because it was made of "three twos."

> Mei: I think I know what he is saying. . . . I think what he's saying is that you have three groups of 2. And 3 is an odd number so 6 can be an odd number *and* an even number.
>
> Ball: Is that what you are saying, Sean?
>
> Sean: Yeah.

Mei said she disagreed. "Can I show it on the board?" She drew ten circles and divided them into five groups of two:



> Mei: Why don't you call *other* numbers an odd number and an even number? What about 10? Why don't you call 10 an even and an odd number?
>
> Sean: (paused, studying her drawing carefully) I didn't think of it that way. Thank you for bringing it up, and I agree. I say 10 *can* be odd or even.
>
> Mei: (with some agitation) What about *other* numbers? Like, if you keep on going *on* like that and you say that *other* numbers are odd and even, maybe we'll end up with *all* numbers are odd and even! Then it won't make sense that all numbers should be odd and even, because if all numbers were odd and even, we wouldn't be even *having* this discussion!

Ofala said she disagreed with Sean because "if you wanted an odd number, usually, like," (and she drew some hash marks on the board) ". . . even numbers are something like it. Even numbers have two in them (she circled the hash marks in groups of two) and also *odd* numbers have two in them——except they have one left":

I called attention to what I referred to as "Ofala's definition for odd numbers," which she restated as "an odd number is something that has one left over." I realized her formulation was, in essence, the formal mathematical definition of an odd number: $2k + 1$. The children tried some experiments with it——with numbers that they expected to work because they already knew them to be odd. Temba tried 3, Betsy tried 21, and Cassandra tried 17. Each time, when they represented the numbers with hash marks and circled groups of two, they found that they had one left over. Later, Riba was still thinking about what Sean had proposed about some numbers being both even and odd. She said that "it doesn't matter how much circles there are——how much times you circle two, it doesn't prove that 6 is an odd number." Ofala agreed.

But Sean persisted with this idea that some numbers could be both even and odd. On one hand, Sean was wrong: Even and odd are defined to be nonoverlapping——even numbers being multiples of two and odd numbers being multiples of two plus one. He was, as Riba pointed out, paying attention to something that was irrelevant to the conventional definitions for even and odd numbers——that is, how *many* groups of two an even number has. On the other hand, looking at the fact that 6 has *three* groups of two and 10 has *five* groups of two, Sean noticed that some even numbers have an *odd* number of groups of two. Hence, they were, to him, special. I thought about how I could treat this as a mathematical invention——and whether I should. I wrote in my journal:

> I'm wondering if I should introduce to the class the idea that Sean has identified (discovered) a new category of numbers——those that have the property he has noted. We could name them after him. Or maybe this is silly——will just confuse them since it's nonstandard knowledge——i.e., not part of the wider mathematical community's shared knowledge. I have to think about this. It has the potential to enhance what kids are thinking about "definition" and its role, nature, and purpose in mathematical activity and discourse, which, after all, has been a major point this week. What should a definition do? Why is it needed?

I thought about the fact that I wanted the children to be learning about how mathematical knowledge evolves. I also wanted them to have experience with what a mathematical community might do when novel ideas are presented. In the end, I decided not to label Sean's claim wrong. Instead, I decided to legitimize his idea of numbers that can be "both even and odd." I pointed out that Sean had invented another kind of number that we hadn't known of before and suggesting that we call these numbers "Sean numbers." He was clearly pleased. The others were quite interested. I pressed him for the definition of Sean numbers:

> *Sean numbers have an odd number of groups of two.*

And, over the course of the next few days, some children explored patterns with Sean numbers, just as others were investigating patterns with even and odd numbers. Sean numbers occur every four numbers——why? If you add two Sean numbers, do you get another Sean number?[7] If a large number ends with a Sean number in the one's place, is the number a Sean number?

Students' learning. Often I must grapple with whether or not to validate nonstandard ideas. Choosing to legitimize Sean numbers was more difficult than valuing unconventional solutions or methods. I worried: Would children be confused? Would Sean numbers interfere with the required "conventional" understandings of even and odd numbers? Or would the experience of inventing a category of number, a category that overlaps with others, prepare the children for their subsequent encounters with primes, multiples, and squares? How would their ideas about the role of definition be affected? I was quite uncertain about these questions, but it seemed defensible to give the class firsthand experience in seeing themselves capable of plausible mathematical creations.

[7]Since a Sean number has an odd number of groups of two, the sum of two Sean numbers will have an even number of groups of two (because odd + odd = even), and so will *never* equal a Sean number.

When I gave a quiz on odd and even numbers, a quiz that entailed some of the kinds of mathematical reasoning we had been using, the results were reassuring. Everyone was able to give a sound definition of odd numbers and to correctly identify and justify even and odd numbers. And, interestingly, in a problem that involved placing some numbers into a string picture (Venn diagram), no one placed 90 (a Sean number) into the intersection between even and odd numbers. If they were confused about these classifications of number, the quizzes did not reveal it.

Dilemmas of respecting children as thinkers. As a mathematics teacher, I am responsible for certain content. My students are supposed to be able to identify even and odd numbers, add and subtract, measure, understand fractions, and much more. Often my problem is to figure out where they are in their thinking and understanding after which I must help to build bridges between what they already know and what there is to learn. Sometimes my problem is that it is very difficult to figure out what some students know or believe——either because they cannot put into words what they are thinking or because *I* cannot track what they are saying. And sometimes, as in the example of Sean numbers, students present ideas that are very different from standard mathematics. The ability to *hear* what children are saying transcends disposition, aural acuity, and knowledge, although it also depends on all of these. And even when you think you have heard, deciding what to do is often a trek over uncharted and uncertain ground. Although Sean was, in a conventional sense, wrong, who could say definitively that he was *wrong*?

Dilemma #3: Creating and Using Community

Classrooms as learning communities (Schwab, 1976) are not a new idea. In my teaching, I am trying to model my classroom as a community of mathematical discourse, in which the validity for ideas rests on reason and mathematical argument, rather than on the authority of the teacher or the answer key (cf. Ball, 1988; Lampert, 1986a). In so doing, I aim to develop each individual child's mathematical power *through the use* of the group. In working our way through alternative ways of approaching and solving problems, we confront issues of shared definition and assumptions, crucial in using mathematics sensibly. My role in this is tricky: Surely I am the one centrally responsible for ensuring that students learn the content of the third grade curriculum. I am also responsible for fostering their capacity and disposition to learn more mathematics and to use it in a variety of life situations. In traditional classrooms, answers are right most often because the teacher says so. As one of my new students explained when I asked why she put a little 1 above the tens column when she was adding, "That's what Miss Brown told us to do whenever you carry." Never mind that, in this case, she should have carried a 2 instead of a 1——the underlying principle was not reason, but the teacher's having said so. I am searching for ways to construct classroom discourse such that the students learn to rely on themselves and on mathematical argument for resolving mathematical sense. The dilemmas inherent in trying to use the group to advance the individual and vice versa, all while keeping one's pedagogical eye on the mathematical horizon, are not trivial.

In the following section, I will use a segment from a lesson on integers. The situation occurred on one of the days we were struggling with the building model and trying to make sense as a community of mathematical thinkers. The vignette spotlights the dilemmas of my role, of authority for knowledge, and of the clarifying/confusing tensions inherent in group discussions——all critical aspects of creating and maintaining a community.

The students were stuck on a problem involving negative numbers. What could it mean to try to do 6 + (-6)? What could be the answer? All of them were convinced that -6 + 6 = 0. This was established by use of Nathan's conjecture (which was actually a theorem, but had not been yet labeled as such):

Any number below zero plus that same number above zero equals zero

I was a little surprised that no one put this together with the commutativity of addition to argue that, if -6 + 6 = 0, then 6 + (-6) would have to equal zero as well. That not one child made this connection was striking, and it reminded me of the shifts we assume in conventional mathematics teaching. When children are introduced to rational numbers, for instance, they are simply supposed to carry their notions about operations with them into this new domain.[8]

Perhaps I might have chosen at this point to pose a challenge: "What if someone in the other third-grade class came over and said, `Nathan's conjecture says that any number below zero plus that same number above zero equals zero and I think you could turn it around because 3 + 6 is the same as 6 + 3 so you can turn Nathan's conjecture around too and so I think that the answer to 6 + (-6) is 0?' What would you say?" This is one strategy I use when the group has entrenched itself in an inadequate or incorrect conclusion or assumption. I did not do this in this case, however. It seemed to me that they were right not to assume that what they knew for positive numbers would automatically hold for negatives. Still, you ask, why not press them a bit? It seemed to me a big step to figure out and reason about the arithmetic of integers and I wanted to let it simmer for a while. I thought, too, I could construct an alternative representation with which they could figure out what made sense.

Recall the children's struggles over this problem of 6 + (-6). Sean had argued that 6 + (-6) should just be 6 "because it wouldn't be able to *do* anything. It just stays the same, it stays on the same number. Nothing is happening." And Betsy had, intuitively, put *two* little paper people on the drawing of the building and moved them *toward* each other until they met——at zero. In both cases, I remained silent, not presenting either child with questions to challenge their solutions. I might have asked Sean, as Riba did, "It says plus six below zero. You're supposed to do *something*. You can't just leave it alone. Or, I might have pressed Betsy, whose conclusion was right but whose reasoning incomplete, "What would you do if it said 6 + (-2)?" Or, "Why don't you put two people on the building and move them toward each other when you add two numbers *above* zero——like 6 + 6?" Instead, however, the other children pressed them.

Betsy: Instead of Sean's, I got zero.

Ball: You'd like to put zero here for 6 + (-6)?

Betsy: Do you want to see how I do it?

Ball: Okay.

Others: Yeah!

---

[8]And, in fact, they do——although often erroneously. Witness the numbers of people who believe that when you multiply two numbers, the product will always be greater than either of the factors.

Betsy: Here. You're here, but you can't go up to 12, because that's 6 plus *6*. So, I say it's just the opposite. It's just 6 minus 6.

Sean: But it says plus, not minus!!!

Betsy: But, you're minusing.

Riba: Where'd you get the minus?

Sean: You should just leave it alone. You can't *add* 6 below zero, so you just leave it. Just say "good-bye" and leave it alone and it is still just 6.

Mei: But this 6 below zero would just disappear into thin air!

Sean: I know. It would just disappear because it wouldn't be able to *do* anything. It just stays the same, it stays on the same number. Nothing is happening.

Betsy: But, Sean, what would you do with this 6 below zero then?

Sean: You just say "good-bye" and leave it alone.

Riba: You can't *do* that. It's a *number*.

Sean: I know, but it's not going down. It's going up because it says plus.

Mei: I think I disagree with Betsy and Sean because I came up with the answer 9.

Ball: Okay, why don't you come and show us how you did that.

(*At this point I did not have a clue about what Mei was thinking*.)

Mei: (reaches up and places one of the little paper people on the building) I start here (at 6 above zero) and then I add 3 to that, because when you go 3 and 3——it's 6. Yeah, and then I got 9, so I think the answer is 9.

Ball: Lucy?

Lucy: Where did the other 3 go then?

Mei: Well, see 'cause it's 3 below zero . . .

Sheena: I know what you're saying.

Mei: So when we put 2 in each group in order to make 1 because it's below zero.

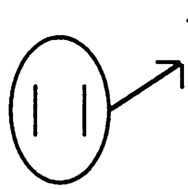
(*I still had no idea what she was doing, but I assumed that if she explained it further, it would make sense in some way*.)

Ball: I don't understand this part——put 2 in each group in order to make 1.

Mei: If we take 6 and add 6 to it, we get 12 above zero, but it's below zero, so——and 3 plus 3 is 6, so we add 3 more to the 6 above zero.

Riba: Mei, is this what you're saying? Three and 3 makes 6. And then you're saying 6 below, and since it's *below*, you have to go up to the 3?

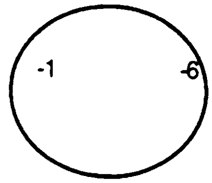
Mei: I'm making, this is one of the numbers, these are two of the numbers below zero (she made two hash marks to represent two of the numbers), and 2 of these equals one (she wrote 1), and if I have about, like——



Betsy: What numbers are these? Can you put the numbers in yet?

Mei: (She paused, wrinkling her face and pondering this request and then drew the following on the board):

Okay, let's see. Two below zero and 3 below zero, and this could be 4 below zero and 5 below zero, equals 1.



And then I have 2 already, and that means you will go up to 8.

(She moved the paper person from the 6 to the 8 on the building.) And then I make 1 more. One below zero and 6 below zero, so there's one more then to go up, and now I end up on 9. (She moved the person one more floor up.)

When she put in numbers at Betsy's request, I realized that Mei hadn't been thinking of particular numbers. She had meant <u>*any*</u> *two numbers below zero would equal 1 (see her first drawing with hash marks) and that you could make three pairs of "below zero" numbers because the problem said "-6." I think Mei was working off a memorized "fact" that "a negative plus a negative equals a positive," something she may have been told by some helpful person. So, taking 6 below zero, and pairing the 6 into three groups of negative numbers (again, look at her drawing with the hash marks), you would get three positive, and would add that 3 to the 6 above zero—— hence, the answer 9.*

Sean: I don't understand what you're trying to say. I thought that you were starting from the 6, plus 6 below, not like 1 plus 1 below zero plus 6 or any other. You're doing all different numbers.

The discussion continued for about 10 more minutes. Ofala said she didn't agree with either Betsy or Mei "because it says plus and you are supposed to be going *up*." Mei replied that if you go up, you end up on the twelfth floor, and that is the answer for 6 + 6, not 6 + (-6). This made sense to Ofala, who then revised[9] her answer. Other children spoke up, either agreeing with one of the presented solutions or questioning one, for example, "If you're going to start with 6, then you have to go up because it's plus?" Sheena objected, "So you're saying that 6 plus 6 equals 12 and 6 plus

[9]We use the term "revise" to denote "changing one's mind," in place of more traditional notions of correcting, fixing, or being wrong.

6 below also equals 12? I don't get it."

Jeannie, who had been quiet all this time, raised her hand. "Jeannie?" I asked. "I'm *confused*," she began slowly. "Betsy said that it is zero, and Mei says that it is 9, and Ofala says that it is 12, and Sean says that it is 6, and I don't know who to believe." I asked her what she had thought when she worked on it before we started the discussion. She said she thought it was zero (the correct answer), "but now I'm not sure."

At this, Cassandra raised her hand. She had changed her mind, listening to the discussion. "I get my person and I started at 6 and I went down 6 more and I ended up at zero." Although this was the end of class and came on the heels of Jeannie's confusion, I still refrained from sealing the issue with my approval. I asked Cassandra why she thought she should go down. "Because it says below zero." Cassandra was now getting the right answer, but her reason was problematic. For instance, when she tries to *subtract* a negative number someday, "going down" will be wrong. This is a problem that arises regularly: Children say things that are true in their current frame of reference, with what they currently know, but that will be wrong in particular contexts later on. For instance, when a first grader announces that 3 is the next number after 2, he is right——in his domain, which is the counting numbers. But, for a sixth grader, considering rational numbers, there *is* no next number after 2.[10] I chose *not* to correct Cassandra's statement for the "mathematical record," believing that such a qualification would pass her and everyone else by anyway. But, as I always do when this happens, I felt a sense of uneasiness and dishonesty.

I knew that some others probably felt as confused as Jeannie did at that moment. She seemed matter-of-fact about her confusion rather than distressed; still, she was confused. And here we were, at the end of the class period. I glanced at the clock and saw that we had five minutes and I made a decision. Moving up by the board, I announced:

> I want everybody to stop talking for one minute now, just think for a minute. I'd like you to find in your notebook where there's an empty space right now. I want you to write down two things. Listen very carefully because you'll have five minutes to do this and I want you to do it carefully. The first thing I want you to write down is what the argument was about that we've been having today, what are we trying to figure out? And then I want you to write down who you agree with most, or if you don't agree with anybody who's up on the board right now, write what you think about this argument. You might not be sure, but write down what you think right now as of October 12, on Thursday at 1:50 p.m. The first thing is, what do you think we've been arguing about, and the second thing is, what do you think

[10]The rational numbers are "infinitely dense," which means that between any two rational numbers, there is another rational number. Between 2 and 2.1, are 2.01, 2.02, and so on. Between 2 and 2.01 are 2.001, 2.002, and so on. Consequently, there is no "next number" unless you specify a context (e.g., the next hundredth).

> about the argument? What do you think the answer should be? And why you think that. There's the first question, what have we been arguing about? The second one is, what do you think and why?

The room was silent as the children wrote intently in their notebooks. Ten out of the 17 children who were in class that day agreed with Betsy, who had argued that 6 + (-6) = 0 (the correct answer). Riba said that "Betsy's ikspachan [*explanation*]" caused her to change her mind. Two agreed with Sean that the answer should be 6. Sheena wrote that she disagreed with Betsy: "betsy is using a minece instead of a plus and its says plus not minece." Three students were not sure. Jeannie said she wasn't "srue hoo to balve" [*sure whom to believe*], although soon thereafter she became convinced that the answer was zero.

Students' learning. We continued to struggle for the next few days with making sense of adding and subtracting negative numbers. I tried to think of better representations for exploring this. When we moved on from negative numbers a week or so later almost every student was able to add and subtract integers accurately if the negative number was in the first position, for example, -5 + 4, or -3 - 8. And many who relied on commutativity or money were able to operate readily with addition and subtraction sentences in any form. This was not a bad achievement.

In addition to learning specifically about operations with integers, what might the students have been learning about community or about the roles of different people——their peers, the teacher, themselves——in their learning? Evidence on this is harder to obtain, but a few snatches from other points, later in the year, help to illuminate some possible learnings. One day, after we had had a particularly long and confusing session on even and odd numbers, I asked the students for comments on the discussion. Sheena commented that "it helps" to hear other people's ideas because "it helps you to understand a little bit more." She gave an example: "I didn't think zero was even *or* odd until yesterday and then someone said it could be even because one below zero and one above zero are both odd, and that made sense."

Mei made a comment that was reminiscent of Jeannie's confusion over the 6 + -6 discussion: "I thought zero was an even number, but from the meeting [the discussion] I got sort of mixed up because I heard other ideas I agree with and now I don't know which one I should agree with." Once again, I saw that children were becoming confused *from* the discussions. But then I asked Mei what she was going to do about this. What she said was significant because of what it revealed about what she may have been coming to understand about herself and about learning mathematics: "I'm going to listen more to the discussion and find out." Both Sheena and Mei, like many of their peers, seemed by midyear to have the sense that they could figure things out together——in group discussions as well as alone.

I asked the class how they felt when, during a discussion, they were arguing a position with which many other people disagreed. Jeannie said that it didn't bother her: "I don't really care how *many* people think [something]. If they changed my mind——if they convince me, then I would change my mind." I asked how they felt when they take a position that no one else in the class is taking. Sean said he "felt fine" about that and that he, too, changed his mind when he was convinced: "I have just changed my mind about 1——that it *is* an odd number." Some children, however, have complained that some of their classmates argue *too* much and that the discussions go on for a long time and "we never find answers."

In general, though, the students seemed to be developing a sense for what they could learn from one another. Riba commented that discussions are helpful because one person may have a good idea when it is taking a long time to figure it out all by yourself. Mei added that, in discussions, "we get ideas from other people." And Sheena said that "it helps us to learn what other people's thoughts are about math because they might teach something new that you never knew before." "Or give us a good example," added Ofala. "Even," said Riba, "maybe the whole class would agree that something was right and only one person in the class would be able to prove that it was wrong."

Dilemmas of creating and using community. Despite evidence that the third graders learn how to learn on their own as well as from one another, there are many days on which I ask myself whether this is time well spent. Take the discussion of 6 + (-6), for example. We spent over half an hour discussing what would be a sensible answer for that one problem. The correct answer was given, but with a problematic explanation. Moreover, two other answers were presented and given equal air time. I did not tell or lead the students to conclude that 6 + (-6) equals zero——by pointing them at the commutativity of addition or at the need for the system of operations on integers to be sensibly consistent. At the end of class, only slightly over half the students knew the right answer. And some misconceptions were floating around——that any negative number plus another negative number equals one positive, for example. Still, the very fact that Mei had carried this misconception into class——based, I suspect, on something someone had probably explained to her about subtracting a negative number——is the kind of thing that keeps me thinking that time spent unpacking ideas is time valuably spent. Too often I have confronted evidence of what children fail to understand and fail to learn from teaching that strives to fill them efficiently with rules and tools.

Two issues lie at the heart of creating and using community in a third grade mathematics classroom: one centered on my role and authority for knowing and learning mathematics, and another centered on balancing confusion and complacency in learning. These two issues are intertwined all of the time: How much should I let the students flounder? Just because it took hundreds of years for mathematicians to accept negative numbers does not necessarily imply that third graders must also struggle endlessly with incorporating negative numbers into their mathematical domain. How much "stuckness" is productive to motivate investigation into the problems that are being pursued? Deciding when to provide an explanation, when to model, when to ask rather pointed questions that can shape the direction of the discourse is delicate and uncertain. Certainly mathematical conventions are not matters for discovery or reinvention——for instance, how we record numbers or what a square is. But that 6 + (-6) must equal zero, or that an even number plus an odd number will always be odd, or that the probability of rolling a 7 with two standard dice is 6/36 are things that children *can* create——through conjecture, exploration, and discussion. Children can also create——as Sean did——new mathematics, new beyond its novelty only for third graders. When is this important?

As the teacher, I know more mathematics than my third graders. There is a lot of mathematics for them to learn. If I understand that 6 + (-6) equals zero and can explain it clearly, it may make sense for me to show them how to deal with adding a negative number and get on with more important things. Nevertheless, orchestrating a classroom community in which participants work together to make sense, as well as developing strategies and ideas for solving mathematical and real-world problems, implies a set of goals that do not exclude——but are not limited to——the children's developing understandings of operations on integers.

The classroom community is often, as the children themselves note, a source of mathematical insights and knowledge. The students hear one another's ideas and have opportunities to articulate and refine or revise their own. Their confidence in themselves as mathematical knowers is often enhanced through this discourse. Still, as the story about the 6 + (-6) dilemma shows, the community can also be a stimulus for confusion. Students with right answers become unsettled in listening to the discussion and sometimes end class uncertain and confused. Are their apparently fragile understandings best strengthened by exposing them to alternative arguments? I worry and I wonder about providing more closure: I open or conclude class discussions with a summary of our problems, conjectures, and puzzlements as often as I open or conclude class with a summary of what we have learned. Are the students learning, from this slow progress to tentative conclusions, that anything goes or that there are no right answers? Or are they learning, as I would like them to, that understanding and sensible conclusions often do not come without work and some frustration and pain——but that they can *do* it, and that it can be immensely satisfying?

Dilemmas of Trying To Be "Intellectually Honest" in Teaching Elementary School Mathematics

In what sense is my practice with third graders "intellectually honest" (Bruner, 1960)? It is honest in its frame——in my concern for students' opportunities to learn about mathematical content, discourse, and community. I try to focus on significant mathematical content and I seek to fashion fruitful representational contexts for students to explore. To do this productively, I must understand the specific mathematical content and its uses, bases, and history, as well as be actively ready to learn more about it through the eyes and experiences of my students. My practice is also honest in its respect for third graders as mathematical thinkers. In order to generate or adapt representations, I must understand a lot about nine-year-olds: What will make sense to them? What will be interesting? How will they take hold of and transform different situations or models? I must integrate the mathematics with children and the children with mathematics. My ears and eyes must search the world around us, the discipline of mathematics, and the world of the child with both mathematical and child filters. And it is from all of these aims and principles that the dilemmas arise that lie at the core of creating a defensible practice: If children believe that zero is not a number, and they are all convinced and agree, what is my role? If all the fraction models I can think of still mislead and distort in some ways, what should I do? When students construct a viable idea that is, from a standard mathematical perspective, reasonable but incorrect, how should I respond?

Dilemmas such as these are not solely the product of the current educational reform rhetoric; many are endemic to teaching (Lampert, 1985). Practice is, after all, inherently uncertain (Jackson, 1986; Lortie, 1975). Still, aiming to create a practice that both honors children and is honest to mathematics clearly heightens the uncertainties. The conception of content in this kind of practice is more uncertain than a traditional view of mathematics as skills and rules, the view of children as thinkers, more unpredictable. Lampert (1985) argues, however, that embracing——rather than trying to resolve——pedagogical dilemmas gives teachers a power to shape the course and outcomes of their work with students. My understandings and assumptions about nine-year-olds equipped me to make decisions about mathematical representation and activity that served their opportunities to learn. Similarly, my notions about mathematics allowed me to hear in the students' ideas the overtures to important understandings and insights.

Because no rules can specify how to manage and balance among competing concerns, teachers must be able to consider multiple perspectives and arguments and to make specific and justifiable decisions about what to do (Lampert, 1986b). Teachers need "the resources to cope with equally weighted alternatives when it is not appropriate to express a preference between them"; they need to be comfortable with "a self that is complicated and sometimes inconsistent" (Lampert, 1985, p. 193). We need to learn more about what are the crucial resources for managing the dilemmas of mathematical pedagogy. That mathematical knowledge is helpful for teachers to have is obvious; the kind and quality of such knowledge is less clear. The same is true for knowledge about students and about learning. Although learning mathematics has traditionally been considered an exclusively psychological matter, other perspectives——linguistic, cultural, sociological, historical——are equally helpful in learning to listen to and interact with children as learners.

In a society in which mathematical success is valued and valuable, reforms that herald a richer understanding and power for students are attractive. But the pedagogical courses are uncertain and complex. How teachers learn to frame and manage the dilemmas of "intellectually honest" practice in ways that do indeed benefit all students is crucial to the promise of such work.

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