BEYOND EXHORTATIONS NOT TO TELL: THE TEACHER'S ROLE IN DISCUSSION-INTENSIVE MATHEMATICS CLASSES

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Pressing for more than curricular change, the current mathematics reforms place an unprecedented emphasis on students' mathematical communication and discourse. These visions of mathematics classrooms imply substantial changes for the teacher's role. Unlike much of their own past experience and practice, teachers are urged to open the classroom to student talk. In small and large groups, students are to present their ideas and solutions, explain their reasoning, and question one another. Accustomed to being the source of knowledge, and doing most of the talking, teachers now must refashion their role in ways that responsibly and effectively shift more authority and autonomy to students. The challenge to the teacher has not gone unrecognized by reformers,[1] and yet, we argue, that the teacher's role in this new vision of classroom learning has been underconsidered. Some people seem to believe that teachers should never tell students anything or that students will learn appropriate mathematics on their own if they are engaged in worthwhile tasks with useful tools and materials. We argue that such a view underestimates the role of the teacher. Few opportunities have existed for careful examination of the complicated decisions and moves that teachers must make in classrooms, and even when such opportunities arise—for example in discussions of videotaped classroom sessions—the linguistic tools available for parsing and analyzing practice often seem too blunt as discussions focus on what the teacher should do, whether they should "tell," "show," "go over," or "let students figure it out on their own."[2]

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[1]Indeed, the NCTM *Professional Standards for Teaching Mathematics* (1991) devotes an entire standard to the "teacher's role in mathematical discourse." In the introduction to this 10-page section, the authors outline the core task of the teacher's role: "to initiate and orchestrate this kind of discourse and to use it skillfully to foster student learning" (p. 34).

[2]One type of occasion in which we have noticed such lacks in our professional discourse has been when we have participated in sessions where teachers, teacher educators, and researchers view and discuss tapes of classroom lessons. Such discussions rarely seem to mine the subtle considerations and moves entailed. Surface characterizations are more common, and appraisals—positive or negative—of the teaching tend to dominate the discussion. We do not have well-developed language or norms for examining teaching. This lack plagues the development of teaching, for individual teachers as well as for the broader community.

This paper originated with this frustration with current discourse about the teacher's role in discussion-intensive teaching. For the past several years, we have been developing and studying teaching practices through our own efforts to teach school mathematics. Ball's work has been at the elementary level, in third grade, and Chazan's at the secondary level, in Algebra I. In our teaching, we have been attempting, among other things, to create opportunities for classroom discussions of the kinds envisioned in the reforms. At the same time, we have been exploring the complexities of such practice. (Others exploring such challenges by teaching include: Hammer 1993; Heaton 1994; Lampert 1986, 1990, 1992; Lensmire 1993, 1994; Roth 1992; Wilson in press; Wong 1995). As researchers who use our teaching as a site for research into, and critique of, what it takes to teach in the ways reformers promote, we have access to a particular “insider” sense of the teacher's purposes and reasoning, beyond that which an outside researcher might have.[3] We felt that more precise and nuanced language could be developed to make distinctions that we noticed in our own efforts to facilitate classroom discussions. This paper represents our work to conceptualize some aspects of the teacher's role in classroom discourse and to contribute tools for the construction, discussion, and analysis of teaching practices. We use two episodes from our own teaching to ground the discussion in a close view of the challenges posed for the teacher's role and follow these descriptions with an analysis of the situations and the pedagogical issues they pose. The paper concludes with an examination of teacher moves aimed at moderating the level and nature of disequilibrium and disagreement.

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[3] At the same time, we aim to be sensitive to the biases and silences which can plague first-person studies of practice. We constantly compare our own firsthand accounts with recordings of classroom sessions, copies of student work, and journal entries written at the end of each session. We engage others—teachers, teacher educators, researchers, policy makers—in viewing tapes and examining students' writing and work; their observations and reactions enhance and expand our perspectives and analyses.

TWO CLASSROOM EPISODES:
CHALLENGES OF THE TEACHER'S ROLE

Algebra I: What to Do About the Zero.

In the first episode, a discussion from Chazan's Algebra I class, students become embroiled in a debate about what to do when averaging a set of pay bonuses where one bonus is $0. In such a scenario, does one count the $0 as a “bonus” at all? Chazan, watching the discussion heat up, grew concerned that it was devolving into little more than a verbal standoff—*Count* the zero! *Don't count* the zero! Seeking a way to resolve, or understand, the students' disagreement productively, Chazan wanted to help the students move their ideas forward. How best to do so was not so clear.

Chazan explains: This class occurred in midwinter. I had been trying to engage the students in considering whether or not it is possible to compute an average without summing the distribution and dividing by the number of numbers (“taking the average”) to expand their sense of what “an average” is and to prepare them for exploring the idea of an “average rate of change.” I had hoped to have students realize that an average bonus depends on the total amount of money available and the number of people—that $5,000 distributed among 10 people yields an average of $500 per person—even if the distribution was $4,991 to one person and one dollar each to the other nine people. This is counterintuitive for my students because it suggests that one doesn't need to know how much each person got and that one doesn't need to “take the average” in order to compute an average. When students think of the average as the result of the procedure of summing and dividing instead of the result of “hypothetically equal sharing,” it is unclear why the word “average” is used to describe an “average rate of change.” Such averages are usually found by subtracting and not adding.

The problem I had given them sketched four different scenarios, each with different distributions of the bonuses to individual employees. The totals of $5,000 and ten employees remained the same across the scenarios. In each scenario, I had asked students to figure out what the average bonus would be. The class discussed the problem for 40 minutes. Things seemed to be going well. We started on the fourth scenario; it was one in which the employer had distributed the bonuses as follows:

$100	$200	$300	$400	$600	$700	$800	$900	$1,000	$0

Jose[4] announced that you could forget about the last person, add up all of the numbers, and then divide by nine or you could divide by ten. He wasn't sure which was best. The discussion went back and forth with students addressing each other. Christin wanted to count the tenth person, giving an average bonus of $500: “If you're going to average this, wouldn't you have to average in the last person because it's still a person? They're just not getting any money. See what I'm saying?”

In order to convince others that the zero should count, Buzz compared the problem of bonuses to computing semester grades. In school, when grades are computed, zeroes on tests or quizzes are counted. However, this analogy only seemed to confuse the other students. Puzzled, Bob pointed out that “when you do our grades, people have different point averages.”

Lynn thought that “you can't really use the zero. It's not standing for anything.” She wanted to divide by 9, arriving at an average bonus of $555.56. Some claimed that the zero should not be counted because it was not really a number anyway. Others didn't want to count the zero because zero dollars is not really a bonus.

Alex:	The zeroes aren't representing anything. They're just representing more people.
Chazan:	They're representing people, but they're not representing—?
Alex:	Money

Jose thought that the zero *could* represent a bonus because it is the “money they [the person who didn't get a raise] *didn't* get.”

Calie, Buzz, and Joe argued that the average bonus should be $500 no matter how the money is distributed. Calie explained, “If you divide 5,000 by 5—oops, by 10—it's going to give you $500 no matter . . .”

The students in this class are on a track which makes it difficult to go to college; they are taking Algebra I—traditionally a ninth-grade course—as tenth-, eleventh-, or twelfth-graders. Although students in this sort of lower-track class are often skeptical about listening to the ideas of others (why listen to others if everyone in your class is there because they are not “good at math”?), on this particular day, I thought they did seem to be listening to one

[4] All names are same-sex pseudonyms. The high school students in Chazan's class selected their own pseudonyms. Ball selected pseudonyms for her third-graders, seeking matches on the basis of language and ethnicity.

another. They did seem to be engaging in the issue and bringing their own experience to bear. I was pleased. Opinion in the class was divided; students were taking turns talking and making references to previous comments.

I was enjoying the discussion and appreciating students' engagement when I began to grow uneasy. I wondered about where the class would go with the disagreement over the zero. Now that the views had been presented, would students be willing to reflect on their own views and change them or would each argue relentlessly for his or her own view? Would they be able to come to some way to decide whether these averages were correct?

My concern stemmed from a desire to have students appreciate what they had accomplished so far and to go farther. From past experience, I knew that students in this class tended to become frustrated with unresolved disagreements and might either turn to me and ask me to tell them who was right and who was wrong or might try to intimidate everyone into agreeing with them. I suspected that in order to feel that the discussion was worthwhile, they would need to feel that their ideas had developed or that they had come to some kind of conclusion or closure—or at least see their way towards some resolution. I wanted to reach this closure in a different way: I wanted students to engage in mathematical reasoning and to decide whether the two answers we'd heard were “correct” or not.

Shifting the Mathematical Focus Away From the Zero

After the discussion of this scenario had gone on for 10 minutes, I decided I had to do something. I considered a range of options. I could have asked different students why they were dividing by 9 or 10. I could have tried to understand how the students who were dividing by 9 saw the problem situation differently from those who were dividing by 10. But I did none of these things. Instead, I decided to ask students to change the focus of the conversation and to think about the question of what the final result ($500 or $555.56) *means*, what it tells us about the situation. From my perspective, the number revealed what each person would have gotten if the total amount in bonuses was equally shared. The two different numbers represented different interpretations of the situation: $500 was how much each person would have gotten if all ten people were to get the same bonus; $555.56

represented the amount that each of the nine people who received bonuses would have gotten if they all received the same bonus while one person received none.[5]

I had two reasons for wanting to focus the conversation on the meaning of the average. My content goal had been to raise the question of whether it is possible to compute an average without summing and to deepen my students' understanding of the concept of "average." Through the discussion, students' ideas might also develop further.

However, I also wanted students to develop confidence in their ability to reason their way to mathematical decisions. One way such decisions are made is through clarification of and reference to first principles. In this case, the basic notion was the meaning of the concept of average. I thought that by thinking about what the average tells us, students might have reason to decide that either $500 or $555.56—or both numbers—were valid answers to the question of the average bonus for the given distribution. So I decided to shift the focus so they could ultimately come back to the question of the zero but with a different perspective.

I changed the topic of the conversation away from the particular set of bonuses and tried to draw students' attention on the question of the meaning of the number that we get as a result of "taking the average":

What I'm thinking is the thing that is hard about this is we have to decide: What do we think an average means? Okay, what do we think the average means? . . . Some people get more than the average, some people get less than the average. This person at 1,000 got a lot more than the average. This person that got 100 got a lot less than the average. These people at 600, they got a little more than the average. So there is a big range—What's Buzz saying when he's saying that 500 is the average?[6]

As I listened to students' responses, I was concerned that they were too vague and that they would not help the class return to the question of the zero productively. They were not saying enough about what the 500 meant.

Rebecca: That's about the amount that everybody's going to get, it's about 500 dollars.

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[5]In this view, an average refers to a situation not as it really is but as it might be reimagined. This kind of reimagining is characteristic of mathematics (O'Connor 1994). It is this reimagining, or hypothetical, or "abstract" (Mokros and Russell 1992), quality of the arithmetic mean which causes much difficulty for students (Mokros and Russell 1992; Pollatsek, Lima, and Well 1981; Strauss and Bichler 1988).

[6]Note that the question is raised in terms of one person's answer. This is an attempt to deal with the ambiguity raised by the different answers.

Bob: It's the number between the highest and lowest amount that people are going to get.

Joe started to explain and then fell back to a description of the procedure for computing the average from a set of data: “Average is . . . you add up all the numbers and you count how many numbers there are, then you divide by that number.” I realized that the students simply did not have the resources—for example, an understanding of division as equal sharing—to deal with the question of what the 500 means. Yet this seemed at the heart of the problem and of the notion of an average.

Returning to the Problem of the Zero

At this point, Christin stepped in and changed the topic back to whether the zero should or shouldn't be counted: “So . . . see . . .you . . .By what he is saying, I think you should add the zero.”

In some ways, Christin's move was exactly the kind of move I would have liked to have a student make. She was taking the result of the class's discussion of the meaning of the 500 and applying it to the controversy about the zero. But they hadn't made much progress on the meaning of the 500. With her assertion, they were headed back to how the average is computed without having come to an understanding of what the final number says.

The discussion took off again. I was intrigued by the level of interest this scenario had generated. Students were talking directly to each other. They argued about whether including the zero was tantamount to including a person.

Rachel: You're supposed to add the zero. Because if you don't, then it's just going to be the average of nine people, and it wouldn't make sense to just cut off the zero. Just totally eliminate it . . .
Victoria: Well, the zero isn't going to count 'cause it doesn't add anything so it doesn't tell you that. It's nothing.

Students were all talking at once at this point. The volume and intensity rose. Some of the students seemed to be ganging up on Victoria.

Alex: . . . 5,000 . . . (inaudible)
Christin: Why would you want to have average of nine people?
Victoria: But the zero doesn't give you ten people. It just adds another . . .

Michael:	Yeah, it does, because ten people are counted.

It was getting still louder. There was a lot of commotion. “Dang!” exclaimed Jane suddenly. Jane seemed surprised that her classmates cared so much about a math problem.

The room was in a commotion. At the same time, I felt that I had learned something new.[7] Although averages are typically computed using the complete distribution, I was finding the argument about a bonus of zero dollars not being a bonus compelling. It started to seem silly to say that one person got a bonus of zero dollars instead of saying that the person didn't get a bonus and shouldn't be considered in computing the average bonus. Before the discussion, I hadn't quite thought about it that way; however, thinking about the arithmetic mean, the average for ten people should be $500. If one wanted to compare the bonuses in two firms, one would certainly want to count the person who isn't getting a bonus.

I suspected that Lynn and the others who were being quiet agreed with Victoria. I intervened to settle the class down, but the argument burst forth again when Joe suggested that the computation you choose depends on whether you think zero is or isn't a number. The comments came quickly, flowing over one another. People had the floor for a short time.

Lynn:	Zero is neutral. It doesn't matter either way if you add it in or not. If the zero . . .
Victoria:	That's what I thought . . .
Jane:	. . . if you add anything by zero, it's going to be the same number.
Chazan:	Okay, now . . .
Joe:	It takes up a place.
Alex:	You need zero to count for the tenth person (inaudible)

Clearly, a large number of students were interested and were participating. For a lower-track class, this session was extraordinary.

At the same time, the discussion did not seem to be helping students progress towards a consensus based on mathematical reasoning. It seemed the class was heading toward an “IS!”/“ISN'T!” kind of argument. There was more argument than reflection and mathematical reasoning. As we leave this episode, the lesson continues. As the teacher, I remained concerned that there was a lot of disagreement, little self-reflection, and no ground for the creation of a consensus based on mathematical reasoning. I worried that, as a result, students'

[7]This learning on Chazan's part is similar to that described in Russell et al. (1994).

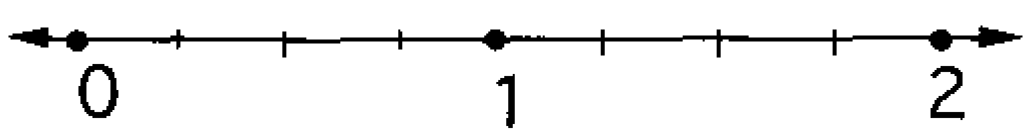
ideas would not develop and they would also not appreciate the achievement which the discussion represented. I wondered how best to help the students use mathematical reasoning to come to some agreements on what would constitute reasonable solutions to this scenario.

THIRD GRADE: LINES VERSUS PIECES

We turn now to Ball's third-grade class. On this day, which occurred in the midst of a unit of work on fractions, her students become satisfied with and convinced by an idiosyncratic way of thinking about the number line. Although the students seem to be agreeing with one another, their conclusion is mathematically problematic. Unlike Chazan's class in which the disagreement seemed to be devolving into a shouting match, this class lacked disagreement. As the teacher, Ball's sense was that she could help students' ideas grow by inserting ideas into the discussion that would challenge and unsettle their conclusion.

The episode occurred in early May. The children had been working on fractions for about two weeks. They had primarily dealt with fractions as parts of wholes, especially as they arise in sharing things and having leftovers—sharing 12 cookies among 5 people, for example. In this work on fractions as parts of wholes, they had explored fractions of a single whole and fractions of groups. For example, not only had they considered 1/4 as one-fourth of one cookie, but they had also considered how 1/4 could mean two cookies if you were talking about one-fourth of eight cookies.

Ball explains: I decided that they needed to extend their work to the number line. This extension seemed important in order to help them develop their understandings of fractions as numbers, not just as parts of regions or groups, and make the systemic shift from the natural or counting numbers to the rationals. On Monday of the third week of the fractions work, I drew a number line from 0 to 2, marked off in fourths, on the board and asked the students to try to figure out what to label the points.

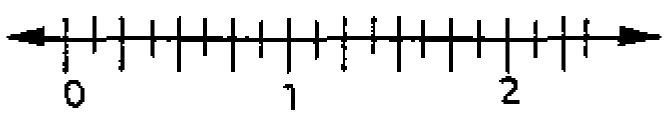


The children worked back and forth between the rectangular-area drawings with which they were comfortable and the less familiar line. For example, some used drawings to prove that 2/4 and 1/2 could both be used to label the point halfway between 0 and 1.

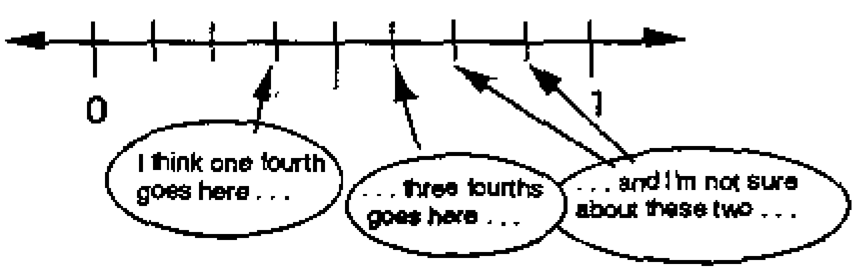
They seemed to be using their part-whole understandings to reason about this new, linear context. Implicitly relying on the distance aspect of the linear model, they made regional models to figure out fractional measures. But they did not make that connection explicit, a fact that emerged the next day when how to understand the points on the number line became an object of disagreement.

Eights on the Number Line

The next day, one of the children asked a question that led us from the more familiar fourths and halves to eights, which we had not yet worked with in any context.



When I asked the class how they could figure out what to call these little lines, Betsy proposed making "cookie" drawings and just cutting them into more pieces. She pointed at the number line and labeled it, apparently visually, without reference to the number of lines.



In her scheme, one-fourth seemed to be the next line to the left of one-half; three-fourths similarly the line to the right of one-half. This schema made sense given that, on other number lines they had seen, they had—at most—labeled three points: 1/4, 2/4 (or 1/2), and 3/4. One-fourth had always been just to the left of 1/2 and 3/4 just to its right. And, in the counting numbers, the position of a particular number was constant—2 always next to 3, 3 always next to 4, and so on.

To figure out the mystery lines—the ones as yet unlabeled—Betsy divided one rectangle into seven pieces and began shading part of it to show how you would figure out what a certain point on the number line was. I grew confused: Why was she using seven? Was it because there were seven little lines *between* the zero and the one? Or was there another reason? What, in her mind, was the correspondence between her rectangle picture and this number line? This, I felt, was a crucial mathematical issue because, if the number line was to represent particular numbers, then the correspondence to another representation (like cookies) could not be arbitrary.

I broke in and asked Betsy whether there were the same number of pieces in her rectangle picture as there were on the number line. Betsy said there were not, that they just needed to have *small* pieces. I paused, surprised.

I often found I could press Betsy in ways that I would not ordinarily push most of my students. A strong and confident child, Betsy was not inclined to follow what I thought merely because I was the teacher. She actually seemed to thrive on disagreement and challenge in situations others might find unnerving. Although Betsy frequently contributed “correct” ideas, she also, at times, argued nonstandard or incorrect ones. I had come to feel that the class often benefited more from Betsy's “incorrect” ideas than from her mathematically standard ones because when I or anyone else challenged her, useful mathematics often became exposed for everyone to work on. I was frankly hoping this could happen here. So I decided to try to challenge her and the rest of the class to figure out a reasonable correspondence between the pictures they drew and the number line that was on the board.

Betsy seemed confused by the question: How many pieces do we *cut* it? She repeated my question, sincerely puzzled. Because I wanted to get the other students more actively involved in Betsy's problem in order to use her confusion as a site for other students' work, I decided to ask a specific question that I hoped would focus the students on the issue. The question had, I thought, only one correct answer. I thought they needed to agree on how they would use drawings as tools to work on this problem.

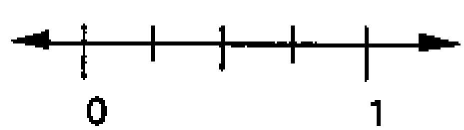
> Ball: Just a second, Betsy. I'd like the *whole* class thinking about it. How many pieces do we need if we want to draw a picture like Betsy's trying to draw? How many pieces *are* there between zero and one right now?

"Six," announced a student. "No, *seven*!" called another. Tory came up and, pointing firmly on the little lines, counted seven sections of the number line. Everyone agreed with her. Seven. There were seven pieces between zero and one.

Provoking Disagreement—Inserting Other Voices

The class was at a key moment: Everyone was agreeing, but what they were agreeing was not *right*. They were certainly right about there being seven *lines* between 0 and 1, but there were eight *pieces*. The number of pieces was what mattered here for making the correspondence between the regional and linear models of fractions. It would not make sense to say that the number line was divided into sevenths if there were seven lines marked. I recalled that they had previously considered and agreed to Sean's conjecture that to make any number of pieces in a drawing, you should make one less *line* than the number of *pieces* you wanted. Yet here they were counting lines, not pieces, but apparently considering them the same thing. Since no one in the class seemed to be connecting yesterday's discussion with this one, I decided to bring it up:

> Ball: Okay, I'd like to show you what you did yesterday, 'cause something you're doing right now—doesn't—isn't the same as what you did yesterday. Stop drawing for a minute. Okay? This is what you did yesterday. I'd like you to think about this for a minute. I'm just going to draw the part between zero and one right now.



Ball: Just look for a minute what you did. When we did this one yesterday, we had three lines and you *didn't* say there were three pieces in there. You said there were *four* pieces. But, today, you're counting differently.

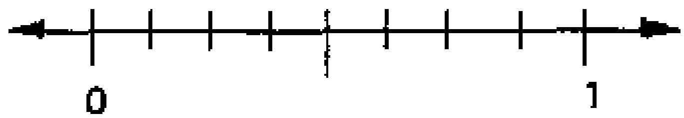
In this instance, I was myself inserting a new voice into the discussion, a voice that I hoped would create some disequilibrium in students' thinking. While this voice was rooted in the students' own work, they were not including it in their discussion.

Often I had found that I could capitalize on the disagreements they had with one another in the course of discussions like this one. As they explored the evidence for competing interpretations or solutions, they could disprove some ideas and come to agree on others. For instance, in a case like this, students would often bring up discrepant interpretations or ideas themselves. Often these controversies among them were sites for mathematical progress. But in this case there was no internal disagreement, no challenge. and their conclusion was wrong.

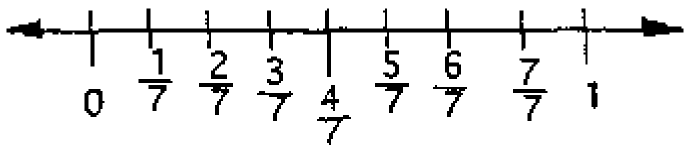
Still, the children did not seem very provoked by my move. No one seemed to sense an inconsistency between what they did yesterday and what they were agreeing upon now. I felt that I needed to press them harder. The way they were agreeing to interpret the number line would make it impossible to connect to other representations of fractions. I continued trying to unsettle their mathematical comfort.

Ball: It looks to me like you skipped a piece. It looks to me like you skipped this piece right here (pointing at the last segment before the 1). The last piece. Because here's (pointing at the spaces between the lines)—Tory counted one piece, two pieces, three pieces, four pieces, five pieces, six pieces, seven pieces, but she didn't count this last piece in here, and I'm curious why.

As with the first move I made, I was trying to play the role that might be played by a student under other circumstances. The mathematical progress of the class drew on its discourse as a community; when the students agreed prematurely, or reached conclusions that were likely to limit their progress, then I could deliberately introduce voices not part of the current conversation. Here I decided that if the class complacently agreed to count seven pieces between 0 and 1 on a number line with seven little lines between those two numbers, that is:



then they might end up agreeing to label that number line as follows:



Thus, 7/7 would seem to be less than 1. Perhaps, by extension, 8/7 would turn out to be equivalent to 1. Four-sevenths would be the same amount as 1/2. Because the numerators are increasing reasonably and in ways that fit the children's prior experience with counting on the number line, this might not seem problematic to them. The labeling would proceed 1, 2, 3, and so on. I could see that in the switch from counting numbers to rational numbers, the need to take into account the meaning of both numerator and denominator mattered here, for the issue was that the first line should be labeled one-*eighth*, not one-*seventh*. I suspected that the students were focused more on the 1 in the numerator than on the denominator. Because the class seemed to be agreeing, I chose to insert comments and observations designed to challenge their agreement. It seemed to me inappropriate to leave this particular conclusion unchallenged. But I remained unsure of what sense they were making of my intervention.

MATHEMATICAL DISAGREEMENTS AND THE TEACHER'S ROLE

Episodes like these are common in our experience. Students share their ideas; they propose solutions; they get stuck and are not sure what they think; they disagree with one another and with their teacher; and they revise their thinking and construct new insights. *Any* discussion holds the potential for discrepant student viewpoints as well as differences between students' views and the views of the mathematical community. In teaching through discussion, these issues cannot be escaped; they are inevitable—and, moreover, essential to students' learning.[8] Thus, managing the differences among ideas in a discussion is one of the crucial challenges for teachers who seek to teach through student exploration and discussion.

Yet how to manage such differences is unclear. When students hold views different from those of the mathematics community, what or who challenges their conclusions and in what ways? Students who are skeptical of school learning may be dismissive of the views of the mathematics community and its norms, while others may change their minds the minute the teacher questions them. In seeking to create democratic classrooms characterized by respect for diverse viewpoints, commitment to learning from students' views (both those that are accepted by the mathematical community and those which are not), and norms for civility, we aim to engage our students with one another and to have them explore, not attack or dismiss, one another's ideas. At the same time, we do not want to present mathematics as mere personal opinion or taste where all opinions are equally valid. Mathematics is a system of human thought, built on centuries of method and invention.[9] Conventions have been developed for testing ideas, for establishing the validity of a proposition, for challenging an assertion. Mathematics has definitions, language, concepts, and assumptions. In the classroom, honoring different viewpoints is crucial. In the mathematics classroom, this also cannot be where it stops.

CURRENT CHARACTERIZATIONS OF THE TEACHER'S ROLE IN CLASSROOM DISCUSSIONS

As suggested before, the current reform movement promotes radical changes in teaching. Traditionally, teaching has been equated with telling; teachers are responsible to present and explain content, to answer questions, and to clear up confusions. The traditional mathematics class session starts with review of previous homework, includes teacher introduction of new material, and culminates with seatwork—guided practice

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[8]This inevitability shows the connection between recent descriptions of mathematicians' ways of working and the activities in which children engage as they learn mathematics. Lakatos (1976), for example, argues that as mathematicians make claims and try to prove them, they uncover new aspects of their initial ideas that require revision. The iterative process of proofs and refutations, and its embracing of conflict as a tool for developing ideas, is an essential quality of mathematical activity similar to the disagreements that arise in our classrooms. Schwab (1978) argues that pedagogical structures should map against disciplinary ones, lest the school work distort essential aspects of the subject matter.

[9]Cobb (1994), examining the interplay of constructivist and sociocultural views of mathematics, argues that a pragmatic mix is required to undergird responsible school mathematics pedagogy.

related to the day's material which prepares students for their homework.

This traditional view is not the image of teaching envisioned in the contemporary mathematics reforms. In reform classrooms, students talk more, teachers show and tell less. Students engage with complex, open-ended problems, in small groups and as a whole class. They present ideas to their classmates and discuss alternative solutions. Through these class discussions, students' understandings develop and progress.

Our experience of this approach to teaching is that, as suggested in *Professional Standards for Teaching Mathematics* (NCTM 1991), the teacher has a complex role to play, shaping the direction and focus of the conversation, guiding appropriate standards of interaction and respect, keeping an eye on the mathematics. Teachers' considerations are complex, their moves subtle. Yet conceptions of the teacher's role often seem focused on what teachers should not do. They should not tell students things; they should not be the source of knowledge. If teachers stay out of the way, the argument goes, students will construct new understandings.[10]

But an exhortation to avoid “telling” seems inadequate as a guide for practice on at least two levels. First, it ignores the significance of context and, as a result, seems to underestimate the teacher's role and suggests that teachers are not supposed to act, regardless of what is going on in the classroom. What does a teacher do when students reach a consensus, but their conclusion is mathematically incorrect? What does a teacher do when a discussion becomes an argument and flashes out of control, hurting feelings? Or what if a discussion focuses on a matter of little mathematical importance? Second, the exhortation to avoid telling is about what *not to do*. It contributes nothing toward what teachers *should do*. While it is intended to allow students a larger role in classroom discussions, it oversimplifies the teacher's role, leaving educators with no framework for the kinds of specific constructive pedagogical moves that teachers might make.[11]

Furthermore, the term “telling” is insufficiently precise. The kinds of “telling” denigrated in reform documents include simply telling students whether their answers are

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[10]Some of these exhortations are grounded in clinical interviewing techniques which have been extremely successful in bringing student thinking to the surface and which have been important in laboratory studies of learning. The question of bringing the sensitivities of clinical interviewing to classroom teaching is examined artfully in *The Having of Wonderful Ideas* (Duckworth 1987).

[11]Smith (1993) focuses on teachers' loss of efficacy when they are left with no clear sense of their role.

right or wrong or giving students correct answers to questions when they have answered incorrectly. This kind of “telling” may not only come in declarative sentences. If the norm (or students' expectation) is that the teacher evaluates every response, teachers can indicate that an answer is incorrect by merely asking a question.

But there are other kinds of telling. Teachers may attach conventional mathematical terminology to a distinction that students are already making. They may return an issue to the classroom “floor”: replaying a comment made by a student or reminding students of a conclusion on which they have already agreed. Teachers may “appropriate”[12] students' comments by rephrasing them as they repeat them to the whole class (Edwards and Mercer 1987). They tell students when they think an utterance was not clear and ask students to make themselves clearer (e.g., “Please say more.” “Why do you think so?”). Finally, teachers also do telling which may not be directly content-related but which may control the focus of a discussion. They tell students to sit down, to come to the board, to listen to others, or give permission to go to the bathroom. They ask the comments after a presentation and may press a particular student by asking whether they agree with a comment that is on the floor.

BEYOND EXHORTATIONS NOT TO TELL: AN ALTERNATIVE CHARACTERIZATION

To conceptualize teaching within a constructivist paradigm, as researchers we need more complex, explicit, and contextualized characterizations of the roles teachers play in discussions and a better language for describing teacher moves. Such theories and descriptions of teaching must be couched in terms of classroom dynamics and their relationship to teacher action and not solely in terms of teacher action. They might serve as a resource for teachers inventing and improvising pedagogical moves and developing a sense of timing in employing these moves.

Our characterization focuses on teacher moves as the product of subtle improvisation in response to the dynamics and substance of student discussion. We aim to capture the relationship between teacher action and the nature and substance of the ongoing discussion, while at the same time challenging the antitelling rhetoric prominent in mathematics education

[12] We mean appropriation in the sense of Cobb's (1994) description of sociocultural theorists' views of the teacher's role. He describes this role as appropriation of students' actions into a wider system of mathematical practices.

reform which focuses acontextually (and sometimes dogmatically) on specific teacher behaviors. Rather than taking a prescriptive view of appropriate teacher moves and style, we argue for a more pragmatic approach in which teacher moves are selected and invented in response to the situation at hand, to the particulars of the child, and to the needs of the mathematics.

"Intellectual Ferment": A Desirable Climate for Learning

Mere sharing of ideas does not necessarily generate learning. For a discussion to be productive of learning, different ideas must be in play: the air is filled with a kind of "intellectual ferment" in which ideas bubble and effervesce. Similar to biological fermentation, this intellectual process cannot be controlled directly but must be guided. However, it can be accelerated by the presence of catalysts. Disagreement—the juxtaposition of alternative ideas—*can* be an important catalyst.[13] As von Glasserfeld notes, "The most frequent source of perturbations for the developing cognitive subject is interaction with others" (quoted in Cobb 1994, p. 14); disagreement with others may cause students to reevaluate and rethink their ideas.[14]

This can even happen for the teacher. In Chazan's class, as a result of the students' argument that people who do not get a bonus should not be counted in computing an average bonus, he rethinks his own position that the average bonus depends solely on the amount of money distributed and not on the particular distribution.

However, fermentation requires a delicate balance: for example, too much heat will kill yeast. Similarly, though disagreement can be a catalyst, it can also shut discussions down. Students' disagreements can lead to confrontation rather than learning. Chazan was concerned that if the discussion about the zero had gone on uncontrolled it might have been settled with fists or intimidation and that mathematical learning would not have occurred. Furthermore, people may vary in their tolerance for and comfort with disagreement.[15] Some

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[13]In a similar vein, Piaget (1952) argues for the importance of disequilibrium as the "motor" for learning and development, though Bruner (1959) takes him to task because disequilibrium does not necessarily result in learning.

[14]This is clearly not the only source. For example, Betsy has unlabeled lines in her drawings which may cause her to rethink her position.

[15]Such variation may have its roots in individual's learning preferences. Some may be rooted in group-related differences—such as gender or culture.

students may feel uncomfortable with disagreement (e.g., Lampert, Rittenhouse, and Crumbaugh 1994) and may retreat. Thus, in our view, during discussions, one of the teacher's roles is to support and sustain intellectual ferment by monitoring and managing classroom disagreement.

Managing Disagreement as a Resource for Student Learning: Three Considerations

In both classroom episodes above, keeping an eye on and helping to stimulate disagreement describes one aspect of the teacher's role. However, the two differ markedly in the kind of disagreement they illustrate and the kind of challenge they pose to the teacher. Chazan is concerned with unproductive *disagreement*—disagreement unaccompanied by reflection (Yackel 1994). In the episode from the Algebra I class, students are heading toward open and entrenched conflict of the IS/ISN'T! variety (O'Connor 1994). Chazan is concerned with how to get the students more thoughtfully focused on the issue. He worries, too, about how to shape and sustain the work in productive directions. And he worries about the students seeing value in even having this discussion.

Ball encounters unproductive and similarly unreflective *agreement* among students; the disagreement is between students and the mathematical community. In the third-grade episode, the issue seems more one of students' mathematical development; students are agreeing on an incorrect method for labeling the number line. Her predicament was: How can a teacher help the group return to their examination of an issue once they seem to have reached consensus?

Discussions are complex intellectual and social events. Diverse students, the relationship among them, their emergent mathematical ideas, the curriculum, the clock—all of these and more interact as a class discussion evolves. If teachers' moves must be constructed in context and seek to create ferment in subtle response to the elements of the specific discussion at hand, what considerations about the context could influence teachers' decisions about action in a discussion? We suggest three sets of considerations: one relies on an appraisal of the mathematics at hand, a second deals with the direction and momentum of the discussion, and a third focuses on the nature of the social and emotional dynamic.

Mathematical Value in Relation to Students. A primary element has to do with the mathematics under discussion: Is it important? Does it have long-term implications for students' learning? Do students currently have the resources for developing the material, or could they reach it meaningfully with some help? In the first episode, it seemed important to Chazan that students move beyond their calculational focus that "taking the average" necessarily requires summing and dividing. Doing this, he thought, would help them consider more deeply the meaning of an "average." The third-graders in Ball's class were inventing a way to label the number line that departed from the necessary correspondence between regional and linear interpretations of fractions. It would stand alone and satisfy them now, but Ball was concerned that it would fragment their developing understanding of fractions. She believed that they could integrate their ideas about part-whole relationships with their newer ideas about fractions as numbers between whole numbers.

Direction and Momentum. A second element focuses on the movement of the students' discussion. Class discussions must have a degree of liveliness, an engaging pace that promises progress and worth. Is it unfolding in a way that promises development, or does it appear to be bogging down? Or is it too hard? The intellectual pace of a discussion can become too steep at times. There can be a need for discussions to "rest," allowing more people in to comment and to consolidate prior work. To do this may mean effecting a plateau in the conversation to include a wider range of responses, giving many students a chance to give a "correct" answer. Alternatively, discussions may lose momentum, bogging down with little challenge. At such moments, the teacher may insert a question or shift the task in a way designed to increase the incline of the intellectual work. Ball tried without a lot of success to steepen the challenge by reminding the students of ideas they had which seemed in conflict with their current ones. In contrast, in Chazan's class, things were not losing momentum. However, Chazan was worried that the direction—seemingly toward simple position-taking—was not likely to produce helpful progress, and he attempted to redirect the work around a different question, to change the direction and focus of the discussion.

Social and Emotional Tone. A third category of concern is less cognitive and less about the intellectual nature of the work than either of the first two. Discussions can become personally unpleasant or they can be respectful and sensitive. Students may grow frustrated with one another, grow impatient, or they may withdraw. They may be engaged, attentive, focused. A direct conflict may be brewing. They may be helpfully building on one another's ideas. The social and emotional barometer of the class is crucial in appraising the degree of ferment and in judging what to do next. Chazan, in his class, worried on this particular day that students were heading for an unhelpful standoff likely to veer increasingly from mathematical to social territory as students converged to ally themselves against Victoria. Ball, in using Betsy's incorrect picture, watched closely to make sure that she was not pushing Betsy too hard in front of her peers or that people were not heading to unite against Betsy as a result of the teacher's challenge.

Making Mathematical Insertions: Telling to Manage Disagreement

In the two episodes above, while taking these three considerations into account, the teacher tried to stimulate, manage, and use disagreement as a resource for the creation of intellectual ferment. In neither case did they simply tell students the “correct” answer. Chazan did not show that $500 was the average bonus; Ball did not show students the “correct” way to label eights on a number line. In both cases, they sought ways to sustain an intellectual process, to have students continue to work on their ideas. Still, neither was passive, staying back while students continued or asking generic neutral questions such as “What do others think?” or “Can you say more about what you were thinking?” Both teachers contributed to the conversation by inserting substantive mathematical comments. We hold this to be a kind of “telling,” a giving of resources, a steering, a provision of something intended to contribute to and shape the discussion.

In each case, the teacher made a substantive mathematical insertion—by making a comment or asking a question. Each introduced mathematics into consideration which up until that point was not part of the conversation. Chazan tried to move the students away from the specific problem of how to calculate an average to the more basic issue of what an average *means*. Knowing that in mathematics as well as in the mathematics classroom definitions are crucial, he posed a question intended to change the focus of their discussion and, hence, their work: “What's Buzz saying when he's saying that 500 is the average?” When students offered formulations that he found vague and insufficient, Chazan challenged students' statements and opened the discussion to others. He actively attempted to manage the mathematical productivity of the discussion.[16]

[16]We are not claiming that the teachers' attempts *work* in any example we show here; rather, we want to illustrate the intricacy of the teacher's role in even *seeking* to manage the productivity of the discussion.

Similarly, Ball, assessing the class climate and work, sought to manage the level and amount of disagreement by bringing new mathematics into the discussion. Twice she sought to introduce other voices designed to provoke students to disagree with themselves. In the first instance, she pointed out that their thinking seemed incongruent with their thinking of the previous day. When they counted a number line divided into eight parts as being in sevenths, she reintroduced something *they* had said in another discussion: "When we did this one yesterday, we had three lines and you didn't say we had three pieces in there; you said there were four pieces . . . but today you're counting differently." Ball's move can be seen as bringing in an idea from the shared class text as a catalyst for reinvigorating the discussion. When this move failed to shift the work—indeed, a student claimed that it was *necessary* to "count differently," Ball drew beyond the students' prior discussions and pushed the class with her objection: "*Why* do you have to 'count differently' today? It looks to me like you skipped a piece . . . and I'm curious why."

In seeking to modulate productively the focus, direction and nature of the discussion, teachers must have a repertoire of ways to add, stir, slow, redirect the class's work. Sizing up a discussion along mathematical, directional, and social dimensions is one task; making moves to shape it is another. Both merit increased attention and more careful parsing in learning to enact—and understand—the teacher's role in managing the complex ferment of mathematical class discussions that can support student learning.

CONCLUSION

> *"The vocabularies that we use [serve as] instruments for coping with things rather than ways of representing their intrinsic nature."*
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> (Cobb 1994, p. 18)

Our exploratory analyses of these two episodes show the value of looking closely at the teacher's moves in relation to classroom context and of the need to sustain, provoke, or temper the degree of ferment among a group of students. These analyses offer one way of examining the teacher's role in leading discussions. Closer study of this role can contribute to the study of the interactive constitution of the discourse in—and, hence, the curriculum of—mathematics classrooms.

Yet, in order to carry out such close study—essential for the complex task of developing what it might mean to teach "for understanding" based on constructivist theories of learning in the spirit of the *Standards*—teachers and teacher educators need intellectual and social resources. These resources include as yet undeveloped tools, images, and ideas as well as contexts and ways of thinking. However, typical patterns of discourse about teaching practice do not support development and invention. All too often, discussions about teaching are reduced to evaluative comments about whether particular teaching is good or bad. The common syntax of "shoulds" and "should haves" distorts practice with a stance of implied clarity. As researcher-teachers, we claim that what is needed is less evaluation and more careful analysis: less embracing or rejecting of particular lessons and more effort aimed at developing understandings of and reasoning about practice.

A discourse supportive of these aims requires both language and stance: language capable of finer distinctions and a stance less aimed at evaluation. For instance, to say merely that the teacher "told" students something is an insufficient description to understand what the teacher did. We need to understand what kind of "telling" it was, what motivated this "telling," and what the teacher thought the telling would do. We need ways of probing the sense that different students make of different teacher moves. Research can contribute to developing language with which subtler descriptions are possible, offering greater conceptual insight and discernment within discourse about practice. In this spirit, we hope that the development of vocabularies for describing the teacher's role in light of constructivist views of learning, which are at the same time sensitive to classroom context, will enhance opportunities for sustained, critical, and insightful discourse among researchers, teachers, and teacher educators about teaching.

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