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# ON MATHEMATICIANS IN CURRICULUM REFORM IN ELEMENTARY MATHEMATICS

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## Center for the Learning and Teaching of Elementary Subjects

The **Center for the Learning and Teaching of Elementary Subjects** was awarded to Michigan State University in 1987 after a nationwide competition. Funded by the Office of Educational Research and Improvement, U.S. Department of Education, the Elementary Subjects Center is a major project housed in the Institute for Research on Teaching (IRT). The program focuses on conceptual understanding, higher order thinking, and problem solving in elementary school teaching of mathematics, science, social studies, literature, and the arts. Center researchers are identifying exemplary curriculum, instruction, and evaluation practices in the teaching of these school subjects; studying these practices to build new hypotheses about how the effectiveness of elementary schools can be improved; testing these hypotheses through school-based research; and making specific recommendations for the improvement of school policies, instructional materials, assessment procedures, and teaching practices. Research questions include, What content should be taught when teaching for conceptual understanding and higher level learning? How do teachers concentrate their teaching to use their limited resources best? and In what ways is good teaching subject matter-specific?

The work is designed to unfold in three phases, beginning with literature review and interview studies designed to elicit and synthesize the points of view of various stakeholders (representatives of the underlying academic disciplines, intellectual leaders and organizations concerned with curriculum and instruction in school subjects, classroom teachers, state- and district-level policymakers) concerning ideal curriculum, instruction, and evaluation practices in these five content areas at the elementary level. Phase II involves interview and observation methods designed to describe current practice, and in particular, best practice as observed in the classrooms of teachers believed to be outstanding. Phase II also involves analysis of curricula (both widely used curriculum series and distinctive curricula developed with special emphasis on conceptual understanding and higher order applications), as another approach to gathering information about current practices. In Phase III, test models of ideal practice will be developed based on what has been learned and synthesized from the first two phases.

The findings of Center research are published by the IRT in the Elementary Subjects Center Series. Information about the Center is included in the IRT Communication Quarterly (a newsletter for practitioners) and in lists and catalogs of IRT publications. For more information, to receive a list or catalog, or to be placed on the IRT mailing list to receive the newsletter, please write to the Editor, Institute for Research on Teaching, 252 Erickson Hall, Michigan State University, East Lansing, Michigan 48824-1034.

## Abstract

The first part of this paper is devoted to a discussion of the previous involvement of mathematicians in curriculum reform projects. Since much has been written on this subject, the second part of this paper reviews only the most notable of these projects, the School Mathematics Study Group, then gives a few general remarks, and briefly discusses the Comprehensive School Mathematics Project.

The third part speculates on the future content of the elementary curriculum and the impact of the mathematical community on the curriculum. Some discussion of the following issues is included: The influence of the computer, an integrated curriculum, the issue of "constant review" in the curriculum, applications, and problem solving.

The fourth part addresses the question of important content. In the author's view the intersection of all suitable curricula is not a suitable curriculum! While the elementary mathematics curriculum is and should remain mostly arithmetic, including estimation, probability, statistics, use of hand-held calculators, and so forth, it should also include other topics such as geometry and spatial visualization, logical deduction, problem solving, and also something on the nature of mathematics as a living and growing subject. Moreover, when a topic is taught it should be developed as a set of ideas coming into play and building up to give a theory of, or meaningful picture of, the topic.

The final section of the paper is a sample unit on area at about the fifth- or sixth-grade level. The unit builds from two basic properties of area, through the question of the area of a rectangle and the question of the area of a right triangle, to the area of triangles, parallelograms, and trapezoids; and, of course, the list of questions need not stop there. The unit does contain some sample exercises, but is not entirely self-contained; presumably the children have a textbook, other discussions of geometry have occurred earlier in the students' studies, and so on.

# ON MATHEMATICIANS IN CURRICULUM REFORM IN ELEMENTARY MATHEMATICS

David E. Blair[1]

## Introduction

This is one of a series of eight reports being prepared for Study 2 of Phase I of the research agenda of the Center for the Learning and Teaching of Elementary Subjects. Phase I calls for surveying and synthesizing the opinions of various categories of experts concerning the nature of elementary-level instruction in mathematics, science, social studies, literature, and the arts, with particular attention to how teaching for understanding and problem solving should be handled within such instruction. Michigan State University faculty who have made important contributions to their own disciplines were invited to become Board of Discipline members and to prepare papers describing historical developments and current thinking in their respective disciplines concerning what ought to be included in the elementary school curriculum. These papers include a sociohistorical analysis of how the discipline should be represented as an elementary school subject, what content should be taught, and the nature of the higher level thinking and problem solving outcomes that should be assessed. This paper focuses on the discipline of mathematics; the other seven papers focus on the disciplines of science, political science, geography, history, literature, music, and art.

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## Mathematicians in Curriculum Reform

There has been input from the mathematical community to the school curriculum through formal participation in curriculum projects and informal "speaking out" on issues. Over the years there have been various curriculum reform programs under the auspices of groups like the Mathematical Association of America (MAA), National Science Foundation (NSF), National Council of Teachers of Mathematics (NCTM), and various universities. These programs themselves have been of more than one type and have served a variety of functions, for example, to improve the content of the curriculum, or to improve the background of elementary school teachers through inservices and summer institutes and so on. A history of these programs up to 1970 can be found in A History of Mathematics Education in the United States and Canada published by the NCTM [9] and an analysis of some of the early programs is given in [8].

Only the School Mathematics Study Group will be discussed in detail; the Comprehensive School Mathematics Project (CSMP) will be discussed briefly, and the others are left to the references just mentioned for the interested reader. SMSG was the most famous of the curriculum reform projects of the 1950s and 1960s and typifies the prevailing concepts of curriculum reform at the time. An important point is the extent to which professional mathematicians have been involved in curriculum matters and how they feel curriculum reforms should be made. When I mentioned to one colleague that I was writing on this subject, she remarked, "Boy, that will be a short paper!" This is not entirely fair, however. Many mathematicians were involved from the start of the big reform projects in the mathematics curriculum in the 1950s, and, while such involvement over the years has not been great or very apparent, mathematicians have often been concerned about the future.

It now appears that more mathematicians may be becoming involved as reported in a very recent set of two articles in the Notices of the American Mathematical Society [2]. Indeed there are natural reasons for mathematicians to become concerned and involved. These include (a) The recognition of an ever increasingly technological and changing society. This increase was already recognized in the aftermath of World War II, even before the impetus of Sputnik (see e.g. [9, pp. 238, 256]). Today mathematicians are still seeing this increase and at the same time seeing a falling number of PhDs graduating in mathematics. (b) Mathematicians are also ordinary people who have families and hence children in school who bring home papers and tell about their experiences. This gives mathematicians a chance to see what is going on in the classroom, noting things they like and dislike; they respond in a variety of ways ranging from teaching their children more things at home to becoming involved in a major reform project. (c) Similarly, mathematicians also teach mathematics -- with their own students in calculus class, for example. Many times in such classes they notice the inadequate background of the students.

## Reform Projects

### School Mathematics Study Group

The most notable program for reform in mathematics education to contain major input from the mathematical community was the School Mathematics Study Group (SMSG). It grew out of post-World War II concern for the general school curriculum, a concern that was heightened in mathematics and the sciences by the launching of Sputnik in 1957; many professional mathematicians felt this concern and became involved in SMSG and other projects. A history of this program up to 1970 can be found in [9, pp. 269-281--see also pp.76-78 for a broader commentary]; SMSG was funded by NSF [9]. While the early years of the program, 1958-1964, were primarily devoted to the curriculum for grades 7-12, by 1964 textbooks for the lower grades were available.

Concerning content in the elementary curriculum of SMSG, the program began from the top down. The original idea was to ensure that graduating seniors were prepared to take calculus as university freshmen and the program began by revising the high school curriculum to try to achieve this. It became evident that mathematics at all levels builds upon earlier foundations and that revision was needed at the elementary level as well. The main content of the K-6 curriculum was still arithmetic, but the attempt was made to teach what was going on in arithmetic instead of simply teaching arithmetic skills. For example, subtraction is the inverse of addition, division the inverse of multiplication and not just two more operations that one needs to learn; $2x+3x = 5x$ because of the distributive law, not because two apples plus three apples equals five apples. The writers of SMSG did not dispute the value of drill as an aid to understanding but assumed that, since there was plenty of drill material already available, they did not need to provide it. The main drawback of SMSG in terms of content was its overformalization; the elementary curriculum paid a great deal of lip service to laws and verbalized a great deal without intellectual payoff. These laws (commutative, associative, distributive, etc.) are among the axioms for a field in abstract algebra, the rational numbers, the real numbers, and the complex numbers being the most common fields. This, together with the fact that teachers and parents were often uncomfortable with this level of abstraction, probably accounts for the decline of SMSG and the so-called "back-to-basics" movement.

In addition to revision of the arithmetic curriculum, SMSG also sought to bring into the elementary school a certain amount of intuitive geometry. With regard to the curriculum in both arithmetic and geometry, SMSG advocated the distribution of content across grade levels. The idea of content across grade levels in SMSG was to introduce ideas and terms early for use later, for example, talking about parallel and perpendicular lines and geometric shapes in the lower grades, so that the student did not enter 10th-grade geometry with no experience with these ideas. Similarly, the purpose of the foundational approach to arithmetic was to provide background for the study of algebra. The matter of problem solving in mathematics, which is one of the main emphases of the Elementary Subjects Center, was not a main feature of SMSG and perhaps, as such, one of its defects. Professor Morris Kline was an outspoken critic of SMSG, primarily on the grounds of its lack of motivation and applications, and of the stimulation for creativity on the part of the student (see e.g., Kline [7] cited in [9]).

The question of the distinction among types of students was not originally part of the SMSG program. The original plan was the development of a curriculum for university-bound students. Supplemental money was later attached for nonuniversity-bound students. Mathematics for these students had tended to be "business math" or "shop math" and SMSG made an effort to develop a more intuitive version of the standard curriculum to be taught over a longer period of time. This program was not entirely successful; with more students going to our universities, many were coming from this group, so that there were, and still are, students entering college with three years of high school mathematics but constituting only one year's worth of traditional algebra. A second criticism of this approach is that, since mathematics is a terse subject to begin with, in order to make it more intuitive, the text writers wrote more for these classes, that is, longer paragraphs to explain things--with the result that students with poorer reading skills now had more to read.

The general thrust of SMSG is no longer with us for a variety of reasons, for example, the need for a balance between computational skills and an understanding of the basic and yet abstract ideas behind them as mentioned above, the lack of applications in the program, the lack of preparation, interest, or ability of the teacher. With regard to the last point, any curriculum is only as good as the teacher who teaches it, whether it be in terms of the teacher's understanding of the material, understanding of the reason it is being taught, or enthusiasm for mathematics. In the 1970s SMSG turned more toward research in the teaching and learning of mathematics and away from its writing program. Also at this time there was a swing away from the foundational approach to arithmetic; a so-called "back-to-basics," meaning more drill, even though SMSG never intended to exclude drill. This became somewhat of a dry period, although of course there was still some activity and now there seems to be a resurgence of activity. Perhaps the major heritage of SMSG was that it brought together many professional mathematicians and teachers of mathematics; the writing teams were quite well-balanced in this respect.

## Remarks

The goal of many reform movements was and is to teach mathematics "as it really is"; but how is that? To SMSG mathematics was viewed as a closed, formal system. Like the axiomatic approach to geometry, one starts with undefined terms, lays down axioms, produces a model to show consistency, and then proceeds to develop the subject, that is, to prove the theorems of the subject. Now try the same thing with freshman algebra! To some extent mathematics is like this; mathematicians do make assumptions and deduce things from them. This, however, is only a view of a formalized final product and it misses the intuitive nature of the subject and the motivation. Mathematics is a growing body of knowledge mathematicians care about: "It is felt that more research mathematics has been done in the last 40 years than in all of previous history" ([3, p. 1]). The Egyptians knew empirically that a triangle with sides of lengths 3, 4, and 5 was a right triangle; the Greeks proved the famous Pythagorean theorem, why should this be the end of the story? In fact it is just the beginning; today, for example, in generalizing the plane with its Euclidean distance formula, we study differentiable manifolds with various Riemannian metrics on them (spaces of general dimension, spheres, tori, etc. endowed with a structure for measuring distances). There are many fascinating and beautiful subjects within mathematics--Euclidean geometry, hyperbolic geometry, complex analysis, nonassociative algebras, partial differential equations--and mathematicians want to know more about them. To do this they ask questions and try to deduce the answers; mathematicians prove theorems and each month the professional journals are full of new ones.

To some "mathematics as it really is" is problem solving, and problem solving is a major focal point of the current program in the Elementary Subjects Center. Of course mathematicians solve problems, but the problems are usually part of a particular theory being developed or furthered by the researcher--even within applied mathematics this is case. To many outside mathematics, its value lies in its application to problems in the outside world and this is what they mean by problem solving. Thus, those outside the discipline feel that application is the aspect that should motivate curriculum development. Certainly, as discussed below, there must be some feature motivating the topics taught, be it an application or some intrinsic feature that makes the topic interesting. The matter of problem solving versus the development of mathematics as a subject, though by no means disjoint, is an issue that needs careful consideration, in my opinion, in any curriculum development that is undertaken. Several of these points will be addressed again in the discussion on the content of the curriculum in the next two sections. In short, mathematics must be taught as a subject to be learned and problem solving as a principal part of the development of the subject, as well as part of the teaching of critical thinking in general.

## Comprehensive School Mathematics Project

Some of the curriculum developments have recognized a broader view of mathematics and, accordingly, endeavored to present mathematics as a reflection of university and professional mathematics. This was the case with the Comprehensive School Mathematics Project, which was the most innovative of the major curriculum projects with its "arrows," "minicomputer," "strings," and so on. CSMP was the brainchild of the Belgian mathematics educator, Frédérique Papy, and her mathematician husband, Georges Papy, and developed in this country by the American Cemrel Center, first in Carbondale, Illinois, and then, in St. Louis, Missouri. The program is still in use, requires teacher training before use in the classroom, and has a lot of positive features. It involves the class in discussion: The children contribute their ideas and work together to solve problems and develop their subject. The children make logical deductions from given premises in an informal context. A description of the program can be found in [4]. In section 2 of [4] there is a nice sample unit on deduction using dots on the board to represent children and red and blue arrows to represent the relations, "You are my brother" and "You are my sister." It is important to note that this is also mathematics. CSMP uses reasoning to solve a problem or to deduce some piece of knowledge rather than stressing the formal aspects of a theory. For the elementary grades this kind of material can be used very profitably.

## Speculation on Future Content

To speculate what the mathematical community will have to say in the future concerning the elementary curriculum is a difficult task , but there are some trends that are worthy of comment and discussion. The main emphasis of the curriculum reform movements of the 1950s and 60s was the "teaching of better mathematics"; the emphasis in recent years has been the "better teaching of mathematics." The future will almost certainly contain further revision of content along with the emphasis on better teaching. For one thing, the computer is here to stay and its influence on society is enormous. One of these influences on mathematics is that the computer raises new types of mathematical questions as well as allowing us to make some progress on questions which were previously untractable; coding theory and the theory of fractals are certainly two current and popular examples of this. Consequently, there will be changes in content in the curriculum, probably at all levels, because of the computer itself and not merely because the computer is a useful tool or a teaching aid. At the very least there will be more discrete mathematics injected into the curriculum.

## Integrated Curriculum

An issue which has been part of curriculum reform in the past and is again a trend which will continue in future curriculum studies is that of an integrated curriculum. Here an integrated curriculum means one in which various ideas in mathematics have a bearing on one another and not that during the course of the school year there should be "a little of this and a little of that," for example, a curriculum which integrates algebra and geometry primarily through analytic methods (use of coordinates, lines as linear equations, etc.) but also including the fact that one has theorems in both--one can learn something by deduction in both subjects. An integrated curriculum is also not intended to mean a curriculum which integrates separate disciplines, though this was to some extent the theme of the curriculum project Minnemast, the Minnesota School Mathematics and Science Teaching Project (see e.g. [9, p. 139]).

Actually, one of the goals of SMSG was to develop, in general, a more integrated curriculum than the traditional one at that time. For example, the United States is one of the few countries to teach a year of plane geometry as a unit in the secondary curriculum. SMSG, or more precisely, the Commission on Mathematics of the College Entrance Examination Board which advised the writing group of SMSG, sought a more integrated approach to geometry with the inclusion of more analytic methods in the course [9, p. 278]; this view did not win out then but it may win in the current round of discussion of these curriculum topics. Consider for example the following statement of Professor P. Hilton in 1983 [5]:

> . . . synthetic proofs of geometrical propositions should continue to play a part in the teaching of geometry, but not at the expense of the principal role of geometry as a source of intuition and inspiration and as a means of interpreting and understanding algebraic expressions. (p. 6)

One of the main criticisms of our current curriculum is that it is too much the same year after year with constant review of the same material, especially in the elementary grades but even in high school algebra, (see e.g. [11]). One colleague suggested that this sends students the wrong message, "Why seriously learn this stuff now; it will come up again later!" Or it gives the impression that what has been studied is all there is to mathematics. In my view the best way to review a concept is to use it in the development of some new ideas, allowing time for the review to take place on the students' part, but not just saying, "Let's review . . . now." For example, the review of fractions could easily take place in the context of introducing some elementary probability theory, teaching some problems on estimation, or giving some applications.

I am reminded of an incident my wife observed in a fabric store. A lady was going to make two items out of the same material; one required $1^3/_4$ yard and the other $1^1/_3$ yard. The lady was dumbfounded as to how to ascertain how much of the material she should buy; this is a nice problem especially with the added constraint that fabric is sold by the 1/8 yard or 1/3 yard (foot). (Why fabric store owners are not clamoring for the metric system I do not know!) Now, I am not advocating this problem as a unit in itself, though it could be used in the introduction of a unit on the addition of fractions or estimation. Estimation and having a sense of the answer to a problem are very important ideas which should be taught in the elementary schools. The point in the above anecdote is not so much the mental ability to add the fractions as it is that one should not be dumbfounded by the problem.

## Applications

Another issue in current and future curriculum development is the growing view, among both mathematicians and mathematics educators, that any subject or topic is best taught in the context of an application, rather than just telling the student, "You need to know this stuff for studying algebra, calculus, or whatever later." Application is meant in the broad sense, including in particular, application to other topics within mathematics. If this view prevails it will require a great revision of the curriculum and realistically one must note that to come up with interesting, appropriate, and nonartificial applications at all grade levels is no easy task. Nonetheless it is important that the study of mathematics be motivated by things of interest to the student and hence materials must be developed in this direction. In addition we should make the curriculum challenging. Professional mathematicians went into mathematics because they found it an interesting and challenging subject. Thus we cannot expect to attract more mathematics majors at our universities by making the subject easier. It is also important to encourage school teachers to develop some of these things themselves or at least to help them become more comfortable with these applications.

Problem solving itself is meant in the broad sense; in particular, the motivation may lie within the subject. There is one thing that problem solving is not--it is not routine work; if some laws of areodynamics are worked out, even if in the context of a wingspan of 50 feet, that is problem solving--to run through it again with a wingspan of 52 feet is not problem solving, unless perhaps some critical phenomenon shows up. Problem solving is a difficult thing to teach; however, it must included in the curriculum and the teaching of it must be done in the context of problems that are meaningful to the student. It is important to point out certain techniques or aspects of problem solving as they occur, such as pattern recognition, "guess and check," use of a model, proof by contradiction, and so on; however, there is no set of methods for all possible problems and there is no substitute for experience. The problem or topic introduced must captivate interest and then the curriculum on that topic must go somewhere and develop some ideas in detail.

It is along these lines that indeed mathematicians can be of great help in curriculum matters by suggesting topics and by ensuring the accuracy of new curricula as they are developed, both in terms of correctness and in terms of how they reflect the discipline. Mathematicians should give considerable thought to the development of key ideas in the subject and thereby, it is hoped, influence the K-12 curriculum; for example, is the first term of calculus the first time a student should encounter the idea of limit? Of course mathematicians who are not directly involved should still be very concerned and can make a valuable contribution though their own teaching of mathematics majors who go into teaching.

## Important Content

The question of what content is most important is again not an easy one, since it is a matter of opinion. In my view the first point to be made is that the intersection of all suitable curricula is not a suitable curriculum! It would be very unwise to single out a minimal core content consisting of "what everybody should know" and then admit the possibility of teaching only that, even in cases of low-achieving students, cases where other constraints leave only so much time for mathematics instruction and cases where the teacher is not strong in mathematics. Thus, while the curriculum in elementary school mathematics will remain mostly arithmetic, this must be broadened, as discussed below, and topics from other areas of mathematics included.

In addition, when a topic is introduced it should go somewhere; there must be the development of ideas. Too often the introduction of problem solving or other topics into the curriculum has been, "Ah, here's a cute one"; some of this is nice but it should not preclude the development of ideas nor the inclusion of problem solving in some natural context. In other words, a few topics must be chosen carefully and treated in greater depth. Even in the sample unit that follows, which ends with the problem of finding the area of a trapezoid in terms of its altitude and the length of its parallel sides--which of course need not be the end of a discussion of area--the point is not that all students must have such a formula tucked away in their brains, but that they develop a set of ideas and use them to solve a problem. However, none of us should adopt an attitude that we can subsequently forget everything we have learned--in life, background never hurts! It would seem that the further the development or building up of a topic goes, the greater the amount that will be retained by the student.

The core of the elementary mathematics curriculum is and should remain arithmetic and the things that go with it in the early grades, counting , place value, measurement, and so forth, but not with an over-emphasis on hand algorithms or fast mental arithmetic--mental arithmetic is good to a point for a number of reasons but speed is not one of them. Children should have considerable experience with hand-held calculators. Then this core must be broadened to include work on problem solving, estimation, probability, and statistics (note how often statistics appear in the newspaper and readers should be able to stop and analyze them) and also broadened to include substantial work on informal geometry and spatial visualization.

With the pressures of our modern age being what they are, education has tended, partly under pressure from parents, to take on an attitude of trying to teach everything in preparation for getting ahead in life, getting into a better university, and so on; it is almost vocational training, albeit for a sophisticated vocation. However, something of the liberal arts including modern topics and the development of critical thinking will better provide students with what they need for their future in a rapidly changing world; actually this has always been the case. Thus, mathematicians' and educators' goals are to give their students the ability to understand the surrounding world as well as to prepare them in the elementary grades for their work in secondary school and the university (K-6 is too early to write someone off as not being university-bound). As part of the curriculum then, problem solving needs to be included. Problem solving has been advocated by many mathematicians over the years, and I would certainly recommend that the reader take a look at George Polya's two volume set Mathematical Discovery [10]. Too often students have wanted a single "sure-fire" method to solve all problems and thus avoid having to think about the problem; some students even believe that such a method exists. Too often students memorize formulas as a way to avoid having to understand the subject. On a recent calculus exam, a student did a dismal job on a spherical coordinates question; she correctly wrote in one corner of her paper $x = \rho \cos\theta \sin\phi$, and so on, but it was clear from her work that she did not know what angle was being referred to by $\phi$.

Also, a goal for professional mathematicians and mathematics educators alike, is to convey the beauty of mathematics in its own right; mathematics is a wonderful subject, independent of its application to science and technology. Indeed, one might be tempted to outline a program of courses designed to prepare the student for the study of mathematics at a university all the way to a PhD to be followed by a life of research in the subject. Of course the fraction of school children who eventually become research mathematicians is minute. Nonetheless, a school curriculum in any subject must be fair to that subject and a sound curriculum in any discipline is, I believe, the best for the student in the long run. Mathematics is not just balancing a checkbook or something used by engineers to build better bridges. In this regard there is one similarity between mathematics and music that could be noted, namely that the beauty of these disciplines lies inherently within the subjects themselves. I have been told that by playing Mozart in a barn, the cows will be more contented and thereby produce more milk--this is clearly not the reason we care about Mozart! Thus, in addition to problem solving, computational skills, spatial visualization, and so on, the curriculum should endeavor to instill in students an appreciation for the subject and, ideally, some of the subject's historical and cultural aspects should be included.

Let me briefly mention the topic of spatial visualization as an important topic for inclusion in the elementary and secondary curriculum; certainly I see weaknesses in this area whenever I teach the several-variable calculus. One instance of the lack of spatial visualization was recently reported by a colleague. On an hour exam my colleague asked the students for the equation of the plane determined by a given pair of parallel lines. Among the responses, one student wrote that two parallel lines did not determine a plane and hence the problem had no solution, while another wrote that there were infinitely many planes through the two parallel lines. More spatial visualization needs to be taught in the elementary and secondary curriculum than there is now. This should also include more than just asking the students how many cubes are in a stack shown in a picture where not all of them are visible from the perspective chosen for the picture. Lines which are not parallel may very well be skew; a sphere is not a circle and the word sphere is not too deep a word for a child to learn; interesting surfaces that occur in soap films and in the architecture of many modern buildings provide a stimulating context for students at all levels. Even the Theorem of Desargues (two triangles' perspective from a point are perspective from a line) is an excellent topic for secondary geometry, especially considering that its proof in space requires only incidence properties as does, subsequently, its proof in the plane when viewed as embedded in space (see e.g. [1, pp. 136-139]).

I do want to insert here a further word about the role of computation in the school curriculum. While mentioned above that there should not be an overemphasis on hand calculations and fast mental arithmetic--and certainly not hours of long division--I am not as opposed to a certain amount of computation as some of my colleagues are. A lot of theorems in many branches of mathematics are proved by reducing some very abstract or geometric ideas to something which can be computed, that is, to a lemma whose proof requires the derivation or manipulation of some equations. For example (and the reader is not necessarily expected to understand this sentence, but simply note that such ideas exist), in studying compact submanifolds of some class of manifolds one might compute the Laplacian of the square of the length of the second fundamental form, obtaining an expression which may be non-negative under certain hypotheses on the curvature of the submanifold and the ambient space; then since the integral of the Laplacian vanishes, the expression obtained vanishes yielding a theorem. Even with the advent of symbol manipulative calculators and computer programs, facility in the area of symbolic computation is still going to be essential. Consequently, I believe that the curriculum in the middle and secondary school should contain a reasonable amount of experience with symbolic computation, though hopefully most of the time in some interesting context.

Finally, while this paper is supposed to be the output of a mathematician, mention should be made of the recommendations of the Institute for Research on Teaching at Michigan State University [6]:

> 1. Teachers should place greater emphasis on the development of conceptual understanding and provide opportunities to apply concepts and skills in formulating and solving math problems.
>
> 2. Fewer topics should be covered in greater depth. Every effort should be made to create the expectation that topics taught are to be learned. This should improve students' achievement in and attitudes toward math.
>
> 3. Math curriculum should be better coordinated across grade levels to decrease the extent to which what is taught one year is repeated. This should help students take math seriously.
>
> 4. Math should be given the status and priority of a subject taught at a regularly scheduled time that is rarely interrupted or preempted by other activities. (p. 4)

In addition, let me make two further recommendations, idealistic as they may be. First, that mathematics consultants be actively involved with classroom teachers, and second, that teachers themselves have more time to devote to preparation, in particular preparation for the teaching of problem solving and other nontraditional topics in the curriculum. As reported in [11] mathematics teachers in the United States have heavy teaching loads by international standards.

## Sample Unit on Area

The following is a brief outline of a sample unit on area. The reason for this choice of sample topic is that it illustrates a number of features of mathematics. First, since mathematics builds on itself, a developmental curriculum is, in my view, preferable to one which attempts to cover a lot of different topics without treating any in depth. The students see how certain ideas are used in the development of other ideas which should indeed make the whole subject more interesting. Mathematics is a subject that begins with axioms or assumptions and studies their implications. Secondly, mathematicians solve problems--by problems we do not mean just numerical ones or applied ones but problems involved in the development of the subject. Finally, mathematicians often think and encourage their students to think geometrically, even when dealing with many algebraic problems. As a result, the study of geometry with the spatial visualization that goes with it is important for students.

The unit below is aimed at the fifth- or sixth-grade level. For each section there is a certain amount of material for the teacher to use with the students, involving them in the discussion. The teacher and students can make models or draw figures to illustrate the various points. At the end of each section there are a couple of exercises for class discussion to which more detail can be added and the teacher would then assign further exercises from the imagined textbook.

The unit also presupposes that students have talked about elementary shapes and configurations in the earlier grades even if the presentation there was only that of visual recognition, for example, perpendicular and parallel lines, shapes like triangles, squares, and rectangles, even quadrilateral is not too sophisticated a word for a child to know. It is also supposed that some geometry has already been included in the curriculum of this grade level at this point, for example, a discussion of the side-angle-side congruence of triangles and the angle sum of a triangle = 180°.

## Sample Lesson Unit

### I.

The concept of area is, of course, a very basic notion in both geometry and analysis. At an elementary level the first key property of area is

Property 1. Congruent polygons have the same area.

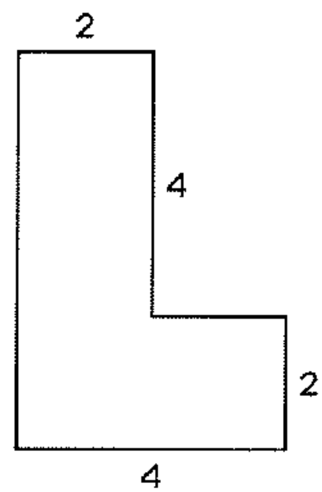
While more sophisticated, the second key property is fundamental to any treatment of area.

> Property 2. If a polygon is decomposed into smaller polygons, the area of the polygon is equal to the sum of the areas of the smaller polygons.

To this we add a unit of area, that is, we designate a certain square as having area one, for example, we might choose a square 1 cm on a side and say its area is 1 square cm.

We are now ready to discuss the area of rectangles whose sides are of integer length with respect to the side length of the unit square. Such a rectangle may be decomposed into squares congruent to the unit square and hence congruent to each other. Thus, using Properties 1 and 2 we can find the area of the rectangle by counting the squares. Now viewing the squares in this decomposition as forming a number of rows with the same number of squares in each row, we see that the area of the rectangle is equal to its length times its width. We now use this idea to define the area of any rectangle: The area of a rectangle is the product of its length and its width.

Exercises I.1. Find the area of the L-shaped region shown.



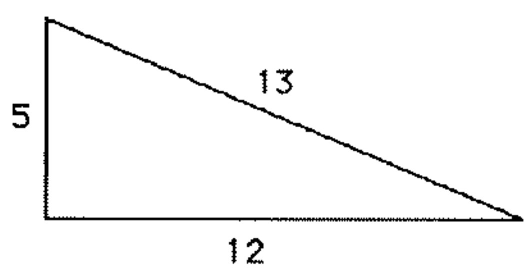

Exercise I.2. How many quarts of paint should one buy to paint a wall 8 feet high and 20 feet long given that one quart of paint will cover 90 square feet?

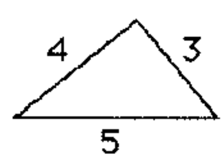
II.

Last time we talked about the area of a rectangle; it is worth bearing in mind that a rectangle is defined as a quadrilateral with four right angles and that a property of rectangles is that opposite sides are congruent--certainly a useful property for tasks, such as laying a brick sidewalk. Let us now consider the problem of finding the area of a right triangle.

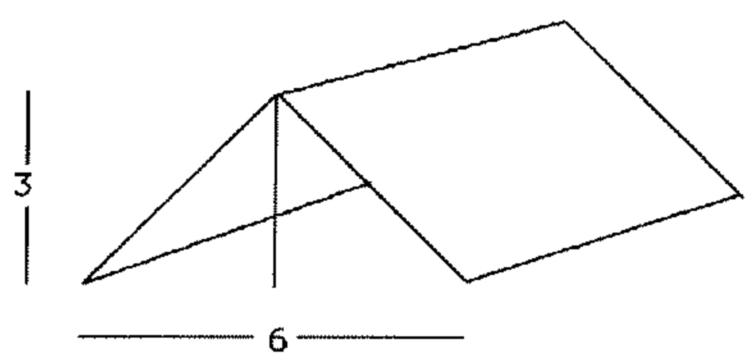
Begin with a rectangle and draw a diagonal; better yet, cut one into two pieces along a diagonal. Are the triangles formed congruent? Yes. The reason is the side-angle-side congruence of triangles (remember a rectangle has four right angles and opposite sides are congruent). By Property 1, the area of each triangle is the same; by Property 2, their sum is the area of the rectangle. Thus, the area of the right triangle is 1/2 the product of the sides forming the right angle. Now the question arises, can any right triangle be achieved in this way, that is, given a right triangle can we construct a rectangle such that when it is cut by a diagonal the pieces are congruent to the given triangle? It may seem obvious that we can simply make a copy of the given triangle and lay it along side the given one with the hypotenuses agreeing and such that complementary angles are matched at the vertices to give the rectangle. This is a subtle point, however--but gives the class an opportunity to use other things they may know. For example, the angle sum of a triangle = 180° (or a straight angle) and right angles have angle measure = 90° (or recall that a right angle is an angle congruent to its supplement).

Exercise II.1. Find the areas of the following right triangles.



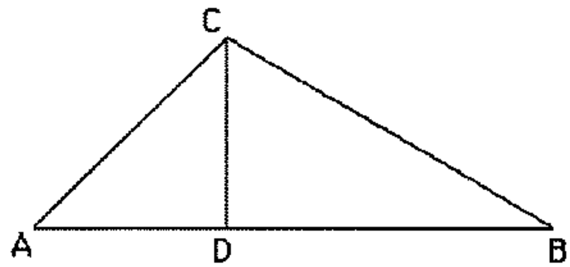




Exercise II.2. The Joneses' children's tent needs new front flaps; they can buy canvas 3' wide or 6' wide. Assuming enough give in the material to account for seaming, how much of which width should they buy?



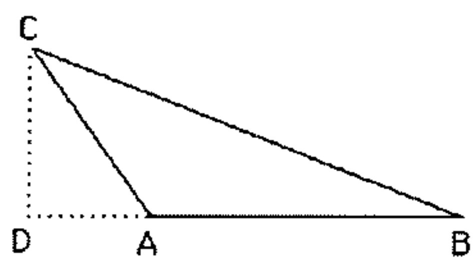
III.

Using what we have learned about area we can now find the area of a general triangle. Given a triangle, label its vertices A, B, C. Now, from the vertex at C drop the perpendicular to side AB and assume for the moment that the perpendicular meets the segment, say at point D.




We then have two right triangles, but we already know how to find the area of a right triangle and by Property 2, the area we seek is the sum of the areas of the two right triangles. Let h be the length of the altitude CD. Thus, the area of the given triangle is 1/2 the product of h and the length of AD plus 1/2 the product of h and the length of DB, but this is 1/2 the product of h and the length of the base AB. Letting b denote the length of the base we see that the area of the triangle is given by $^1/_2bh$.

Now it may have happened that when we dropped the perpendicular from C it did not meet the segment AB but met the line of AB at a point D outside the triangle, say with A between D and B.




We are still, however, in a good position to find the area of $\triangle ABC$ knowing how to find the area of right triangles. Property 2 now gives us that

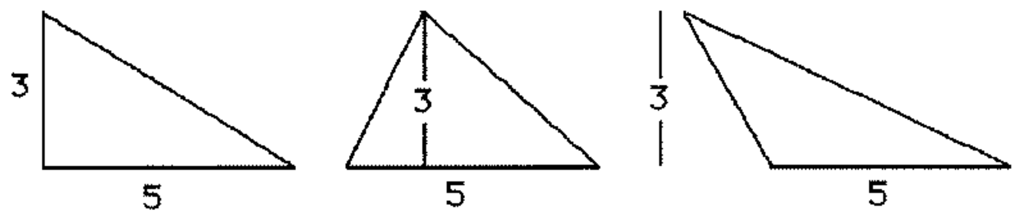
(area of $\triangle BCD$) = (area of $\triangle ABC$) + (area of $\triangle ACD$)

and hence

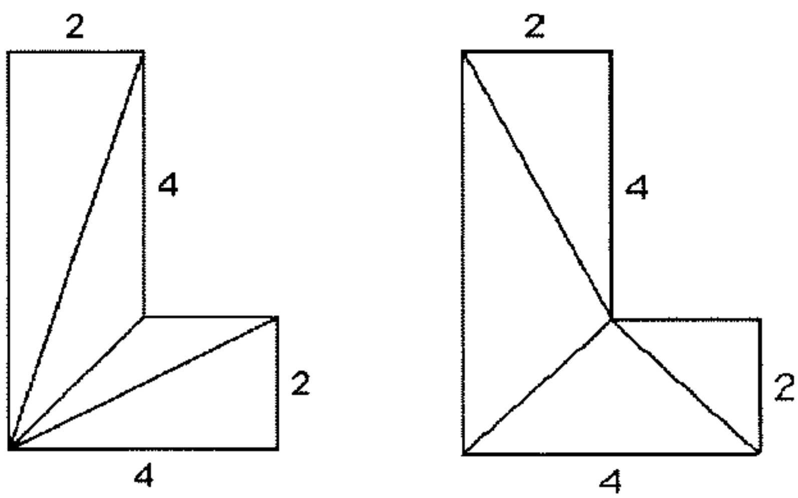
(area of $\triangle ABC$) = (area of $\triangle BCD$) - (area of $\triangle ACD$).

Now, as before, let h be the length of the altitude CD. Thus, the area of the given triangle is 1/2 the product of h and the length of BD minus 1/2 the product of h and the length of AD, but this is 1/2 the product of h and the length of the base AB. Again letting b denote the length of the base we see that the area of the triangle is again by $^1/_2bh$.

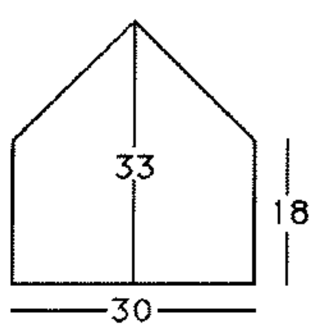
Exercises III.1. Find the area of each of the following triangles.



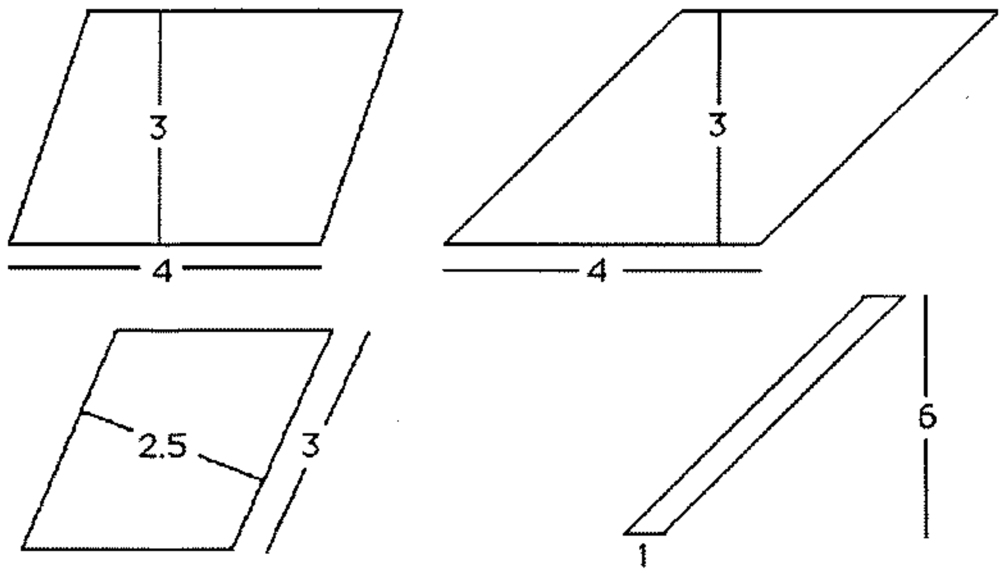

Exercise III.2. Find the area of each of the following triangles.



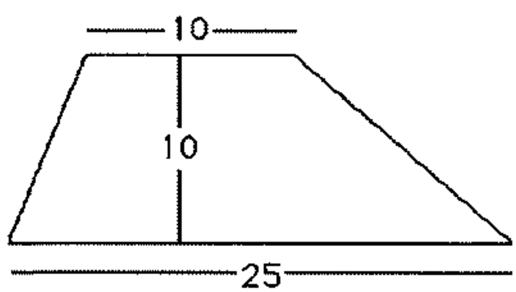

Exercise III.3. Find the amount of paint needed to cover the end of the barn shown, given that one gallon of paint will cover 400 sq ft. If the barn is 40 ft long determine how much paint is needed to paint both ends and both sides.



Exercise III.4. Find the areas of the following parallelograms. Find the area of a general parallelogram in terms of the length of one side and the distance between the lines of that side and its opposite side.




Exercise III.5. Find the area of the trapezoid shown. Find the area of a general trapezoid given the lengths of the parallel sides and the distance between the lines of the parallel sides.



## References


[1] Adler, C. F., Modern Geometry, 2nd Ed., McGraw-Hill, New York, 1967.

[2] American Mathematical Society, Research mathematicians in mathematics education, Notices AMS, 35 (1988) 790-794, 1123-1131.

[3] Commission on Physical Sciences, Mathematical Sciences, Some Research Trends, National Academy Press, Washington, D.C., 1988.

[4] Comprehensive School Mathematics Program, CSMP in Action, Cemrel, St. Louis, 1978.

[5] Hilton, P., Current trends in mathematics and future trends in mathematics education, For the Learning of Mathematics, 4,1 (1984) 2-8.

[6] Institute for Research on Teaching, Elementary math curriculum out of balance, Communication Quarterly, Michigan State University, 10,2 (1988) 1&4.

[7] Kline, M., Math Teaching Assailed as Peril to U.S. Scientific Progress, New York University Alumni News, October 1961.

[8] National Council of Teachers of Mathematics, An Analysis of New Mathematics Programs, NCTM, Washington D. C., 1963.

[9] ____, A History of Mathematics Education, Thirty-second Yearbook of the NCTM, Washington D.C., 1970.

[10] Polya, G., Mathematical Discovery, Wiley, New York, 1962.

[11] Second International Mathematics Study, The Underachieving Curriculum, Stipes, Champaign, 1987.